δ-invariants of Du Val del Pezzo surfaces and K-stability of Fano threefolds

Elena Denisova

Doctor of Philosophy

THE UNIVERSITY OF EDINBURGH

2025

To my parents Anna and Alexey,
my husband Ivan, my son Konstantin,
and my friends Maria and Eva.

Lay Summary

This thesis lies in the field of **Algebraic geometry** — a modern branch of pure mathematics involving spaces with an algebraic structure called **varieties**. There are three fundamental types of varieties: Fano, Calabi–Yau, and General Type varieties, based on how they are curved. This thesis contains the results for **Fano** varieties, whose importance has been highlighted by the work of Birkar, who won a Fields Medal in 2018, and Donaldson, who won a Breakthrough Prize in 2015.

One of the central questions of complex geometry is when the above classes of varieties admit canonical metrics. An important example is a special metric called a **Kähler–Einstein metric**. It was proven by Aubin and Yau that there exists a Kähler–Einstein metric on Calabi–Yau and General Type varieties. However, Fano varieties are more complicated. The main algebraic criterion for the detection of Fano varieties admitting a Kähler–Einstein metric is called **K-stability**, and the surrounding field is one of the most active in pure mathematics. In this thesis, I applied an existing algebraic criterion to detect varieties admitting Kähler–Einstein metrics which were unknown up until now, and unachievable without the results of this thesis.

Abstract

Due to the celebrated solution of the Yau–Tian–Donaldson conjecture, it is now known that a smooth Fano variety admits a Kähler–Einstein metric if and only if it is K-polystable. In **Part I** of this thesis, we study Du Val del Pezzo surfaces (Fano varieties of dimension 2 with explicit "nice" singularities). We present the computation of all the δ-invariants of Du Val del Pezzo of any degree in complete generality. This leads to our description of new examples of K-stable Fano threefolds, undertaken in **Part II** of this thesis. More explicitly, we first present (possibly singular) K-stable examples of Fano threefolds in Families №1.11, №1.12, №1.13, №2.1, №2.2, №2.3, №2.5, which follow from computations in Part I. After that, we find all K-polystable smooth Fano threefolds that can be obtained as blowup of $\mathbb{P}^3$ along the disjoint union of a twisted cubic curve and a line. These are the smooth Fano threefolds in Family №3.12. We conclude by proving the K-stability of smooth Fano threefolds of Picard rank 3 and degree 20 that satisfy very explicit generality conditions. These threefolds are the smooth Fano threefolds in Family №3.5.

Acknowledgements

I am deeply grateful to my family and friends for their constant support and faith in me.

I would like to thank Dr. Theodoros Papazachariou and Dr. Eva Zilber for their valuable feedback, which helped improve the literary quality of this thesis.

I would also like to thank the wonderful mathematicians who taught and guided me along the path that led to this moment:

I am especially grateful to my PhD supervisor, Prof. Ivan Cheltsov, for his willingness to share his deep expertise and for his continuous support. The knowledge he imparted has had a profound impact, both on my thinking and on how I navigate the world of academia. I also thank my second supervisor, Prof. Milena Hering, for her valuable guidance.

I am thankful to my collaborators — Prof. Hamid Abban, Prof. Kento Fujita, Prof. Anne-Sophie Kaloghiros, Prof. Jesus Martinez-Garcia, Prof. Ruadhaí Dervan, Dr. Theodoros Papazachariou, Dr. Erroxe Etxabarri-Alberdi, and Dongchen Jiao — for helping me grow as a mathematician through our joint work.

I would like to thank my master's advisor, Prof. Andrea Fanelli, for his support and for introducing me to birational geometry. I am also grateful to Prof. Yuri Bilu, my tutor at the University of Bordeaux, and Prof. Peter Stevenhagen, my tutor at the University of Leiden, for their invaluable advice throughout my master's degree.

I thank my bachelor's advisor, Prof. Alexei Gorinov, who was always patient and supported my desire to pursue an academic career. I also want to acknowledge Prof. Ossip Schwarzman and Prof. Evgeny Feigin, who supervised my first academic projects.

I am also indebted to my high school teachers — Rafail Gordin, Prof. Petr Sergeev, Dr. Alexey Elagin, and Sergey Shashkov — who laid the foundation of my knowledge and introduced me to the world of academic mathematics.

Finally, I gratefully acknowledge the University of Edinburgh for financial support through the School of Mathematics Studentship (funded via EPSRC DTP).

Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

Elena Denisova

Contents

Lay Summary **iii**

Abstract **iv**

Acknowledgements **v**

Declaration **vi**

Figures and Tables **ix**

1 Introduction **1**

I δ-invariants of Du Val Del Pezzo surfaces **11**

2 δ-invariants of Du Val Del Pezzo surfaces **12**

2.1 History and known results 12

2.2 Proof of Main Theorem via Kento Fujita's formulas 15

3 Du Val del Pezzo surfaces of degrees 7 and 8 **17**

3.1 Del Pezzo surface of degree 8 with $\mathbb{A}_1$ singularity 17

3.2 Del Pezzo surface of degree 7 with $\mathbb{A}_1$ singularity 18

4 Du Val Del Pezzo surfaces of degree 6 **21**

4.1 General results for degree 6 21

4.2 Finding δ-invariants for degree 6 30

5 Du Val Del Pezzo surfaces of degree 5 **33**

5.1 General results for degree 5 33

5.2 Finding δ-invariants for degree 5 49

6 Du Val del Pezzo Surfaces of Degree 4 **52**

6.1 General results for degree 4 52

6.2 Finding δ-invariants for degree 4 75

7 Du Val del Pezzo Surfaces of Degree 3 **82**

7.1 General results for degree 3 82

7.2 Finding δ-invariants for degree 3 114

8 Du Val del Pezzo Surfaces of Degree 2 123
8.1 General results for degree 2 124
8.2 Finding δ-invariants for degree 2 169

9 Du Val del Pezzo Surfaces of Degree 1 192
9.1 Finding δ-invariants for degree 1 194

II K-stability of Fano Threefolds 247

10 First Examples of K-stable Fano Threefolds 248
10.1 K-stable Fano Surfaces 248
10.2 First examples of K-stable Fano Threefolds 249
10.2.1 Family №1.13 (Del Pezzo Threefolds of degree 3) 250
10.2.2 Family №2.5 251
10.2.3 Family №2.2 252
10.2.4 Family №1.12 (Del Pezzo Threefolds of degree 2) 253
10.2.5 Family №2.3 254
10.2.6 Family №1.11 (Del Pezzo Threefolds of degree 1) 255
10.2.7 Family №2.1 256

11 K-stability classification for Family №3.12 258
11.1 Calabi Problem for Family №3.12 258
11.2 Preliminary results 259
11.3 Geometry of Fano Threefolds in Family №3.12 260
11.3.1 Basic properties 260
11.3.2 Construction of R 262
11.3.3 $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ - action on X 262
11.3.4 Mori Cone $\overline{\mathrm{NE}(X)}$ 268
11.3.5 Cone of Effective Divisors $\mathrm{Eff}(X)$ 269
11.4 Proof of the Main Theorem (Family №3.12) 270

12 K-stability in Family №3.5 283
12.1 Calabi Problem for Family №3.5 283
12.2 Proof of the Main Theorem (Family №3.5) 285
12.3 Polarized δ-invariants via Kento Fujita's formulas 288
12.3.1 Du Val Del Pezzo surface of degree 4 with $\mathbb{A}_1$ singularity 289
12.3.2 Du Val Del Pezzo surface of degree 4 with $2\mathbb{A}_1$ singularities 300
12.3.3 Du Val Del Pezzo surface of degree 4 with $\mathbb{A}_2$ singularity 311

Bibliography 314

Figures and Tables

Figures

3.1 Dual graph: $(-K_S)^2 = 7$, singularity $\mathbb{A}_1$ 18

4.1 Dual graph: $(-K_S)^2 = 6$ for a general point 21
4.2 Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = 1$ 23
4.3 Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = \frac{9}{10}$ 23
4.4 Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = \frac{9}{11}$ with $\tau(A) = 2$ 24
4.5 Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = \frac{9}{11}$ with $\tau(A) = 3$ 25
4.6 Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = \frac{4}{5}$ 26
4.7 Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = \frac{3}{4}$ with $\tau(A) = 3$ 26
4.8 Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = \frac{3}{4}$ with $\tau(A) = 2$ 27
4.9 Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = \frac{9}{14}$ 28
4.10 Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = \frac{3}{5}$ 29
4.11 Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = \frac{1}{2}$ 29
4.12 Du Val del Pezzo surfaces with $(-K_S)^2 = 6$ 30

5.1 Dual graph: $(-K_S)^2 = 5$ for a general point 34
5.2 Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{15}{13}$ 35
5.3 Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = 1$ 36
5.4 Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{30}{31}$ 37
5.5 Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{15}{16}$ 38
5.6 Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{15}{17}$ with $\tau(A) = 2$ 38
5.7 Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{15}{17}$ with $\tau(A) = 3$ 40
5.8 Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{15}{19}$ with $\tau(A) = 3$ 40
5.9 Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{15}{19}$ with $\tau(A) = 2$ 41
5.10 Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{10}{13}$ 42
5.11 Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{3}{4}$ 43
5.12 Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{5}{7}$ 43
5.13 Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{30}{43}$ 44
5.14 Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{9}{13}$ 45
5.15 Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{15}{23}$ 46
5.16 Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{3}{5}$ 46
5.17 Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{5}{9}$ 47
5.18 Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{6}{11}$ 48

5.19 Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{3}{7}$ . . . 48
5.20 Du Val del Pezzo surfaces with $(-K_S)^2 = 5$ . . . 50

6.1 $(-K_S)^2 = 4$, a general point on a (-1)-curve . . . 53
6.2 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{6}{5}$ . . . 55
6.3 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{8}{7}$ . . . 56
6.4 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{9}{8}$ . . . 57
6.5 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{12}{11}$ . . . 58
6.6 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{24}{23}$ . . . 59
6.7 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = 1$ with $-K_S - vA$ nef on $[0,1]$ . . . 60
6.8 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = 1$ with $-K_S - vA$ nef on $[0,2]$ . . . 61
6.9 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{6}{7}$ with $\tau(A) = 2$ . . . 62
6.10 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{6}{7}$ with $\tau(A) = 3$ . . . 63
6.11 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{24}{29}$ . . . 64
6.12 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{9}{11}$ . . . 65
6.13 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{4}{5}$ . . . 66
6.14 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{3}{4}$ with $\tau(A) = 2$ . . . 66
6.15 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{3}{4}$ with $\tau(A) = 3$ . . . 67
6.16 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{3}{4}$ with $\tau(A) = 4$ . . . 68
6.17 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{2}{3}$ . . . 69
6.18 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{24}{37}$ . . . 70
6.19 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{9}{14}$ . . . 71
6.20 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{6}{11}$ . . . 71
6.21 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{1}{2}$ with $\tau(A) = 4$ . . . 72
6.22 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{1}{2}$ with $\tau(A) = 5$ . . . 73
6.23 Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{3}{8}$ . . . 74
6.24 Du Val del Pezzo surfaces with $(-K_S)^2 = 4$ (pic. 1/3) . . . 76
6.25 Du Val del Pezzo surfaces with $(-K_S)^2 = 4$ (pic. 2/3) . . . 76
6.26 Du Val del Pezzo surfaces with $(-K_S)^2 = 4$ (pic. 2/2) . . . 77

7.1 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{27}{17}$ . . . 83
7.2 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{3}{2}$ . . . 85
7.3 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{54}{37}$ . . . 86
7.4 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{36}{25}$ . . . 87
7.5 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{10}{7}$ . . . 88
7.6 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{27}{19}$ . . . 88
7.7 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{9}{7}$ . . . 90
7.8 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{6}{5}$ with $\tau(A) = \frac{3}{2}$ . . . 91
7.9 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{6}{5}$ with $\tau(A) = 2$ . . . 93
7.10 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{27}{23}$ . . . 94

7.11 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{9}{8}$ . . . 95
7.12 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{18}{17}$ . . . 95
7.13 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = 1$ with $\tau(A) = 2$ . . . 96
7.14 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = 1$ with $\tau(A) = 3$ . . . 97
7.15 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{18}{19}$ . . . 98
7.16 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{27}{29}$ . . . 99
7.17 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{12}{13}$ . . . 100
7.18 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{9}{10}$ . . . 101
7.19 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{6}{7}$ . . . 102
7.20 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{9}{11}$ . . . 103
7.21 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{18}{23}$ . . . 104
7.22 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{27}{35}$ . . . 104
7.23 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{3}{4}$ . . . 105
7.24 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{9}{13}$ . . . 106
7.25 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{2}{3}$ with $-K_S - vA$ nef on $\left[0, \frac{3}{2}\right]$ . . . 107
7.26 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{2}{3}$ with $-K_S - vA$ nef on $[0,1]$ . . . 108
7.27 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{3}{5}$ with $\tau(A) = 4$ . . . 108
7.28 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{3}{5}$ with $\tau(A) = 3$ . . . 109
7.29 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{9}{19}$ . . . 110
7.30 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{6}{13}$ . . . 111
7.31 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{3}{7}$ . . . 112
7.32 Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{1}{3}$ . . . 113
7.33 Du Val del Pezzo surfaces with $(-K_S)^2 = 3$ . . . 116

8.1 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{45}{16}$ . . . 124
8.2 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = 2$ . . . 125
8.3 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{15}{8}$ . . . 126
8.4 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{24}{13}$ . . . 127
8.5 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{9}{5}$ . . . 129
8.6 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{72}{41}$ . . . 130
8.7 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{12}{7}$ . . . 131
8.8 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{18}{11}$ . . . 132
8.9 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{60}{37}$ . . . 133
8.10 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{36}{23}$ . . . 134
8.11 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{3}{2}$ . . . 135
8.12 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{9}{7}$ . . . 137
8.13 Picture: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{6}{5}$ (intersection of two (-2)-curves) . . . 139
8.14 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{6}{5}$ . . . 140
8.15 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{36}{31}$ . . . 141

8.16 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{8}{7}$ . . . 143
8.17 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{60}{53}$ . . . 144
8.18 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{9}{8}$ with $\tau(A) = \frac{5}{3}$ . . . 145
8.19 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{9}{8}$ with $\tau(A) = \frac{3}{2}$ . . . 146
8.20 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = 1$ . . . 147
8.21 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{12}{13}$ . . . 149
8.22 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{9}{10}$ with $-K_S - vA$ nef on $\left[0, \frac{4}{3}\right]$ . . . 150
8.23 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{9}{10}$ with $-K_S - vA$ nef on $[0, 1]$ . . . 151
8.24 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{6}{7}$ with $-K_S - vA$ nef on $\left[\frac{3}{2}, 2\right]$ . . . 152
8.25 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{6}{7}$ with $-K_S - vA$ nef on $[0, 1]$ . . . 152
8.26 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{4}{5}$ with $\tau(A) = 2$ . . . 153
8.27 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{4}{5}$ with $\tau(A) = \frac{5}{2}$ . . . 154
8.28 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{3}{4}$ with $\tau(A) = 3$ . . . 155
8.29 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{3}{4}$ with $\tau(A) = 2$ . . . 157
8.30 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{3}{4}$ with $\tau(A) = \frac{5}{2}$ . . . 158
8.31 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{3}{5}$ . . . 158
8.32 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{4}{7}$ . . . 159
8.33 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{9}{16}$ . . . 160
8.34 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{1}{2}$ . . . 161
8.35 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{3}{7}$ . . . 162
8.36 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{3}{8}$ . . . 163
8.37 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{3}{10}$ . . . 164
8.38 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) \geq \frac{384}{209}$ . . . 164
8.39 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) \geq \frac{49}{27}$ . . . 165
8.40 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) \geq \frac{216}{121}$ . . . 166
8.41 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) \geq \frac{75}{43}$ . . . 167
8.42 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) \geq \frac{32}{19}$ . . . 167
8.43 Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) \geq \frac{27}{17}$ . . . 168
8.44 Du Val del Pezzo surfaces with $(-K_S)^2 = 2$ (pic. 1/5) . . . 173
8.45 Du Val del Pezzo surfaces with $(-K_S)^2 = 2$ (pic. 2/5) . . . 174
8.46 Du Val del Pezzo surfaces with $(-K_S)^2 = 2$ (pic. 3/5) . . . 175
8.47 Du Val del Pezzo surfaces with $(-K_S)^2 = 2$ (pic. 4/5) . . . 176
8.48 Du Val del Pezzo surfaces with $(-K_S)^2 = 2$ (pic. 5/5) . . . 176

9.1 Picture: $(-K_S)^2 = 1$, $\mathbb{A}_1$ singularity (nodal) . . . 195
9.2 Picture: $(-K_S)^2 = 1$, $\mathbb{A}_1$ singularity (cuspidal) . . . 196
9.3 Picture: $(-K_S)^2 = 1$, $\mathbb{A}_2$ singularity (nodal) . . . 197
9.4 Picture: $(-K_S)^2 = 1$, $\mathbb{A}_2$ singularity (cuspidal) . . . 199
9.5 Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_3$ singularity . . . 200

9.6 Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_4$ singularity 201
9.7 Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_4$ singularity, $\delta_P(S) = \frac{15}{11}$ 202
9.8 Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_5$ singularity 205
9.9 Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_5$ singularity, $\delta_P(S) = \frac{6}{5}$ 205
9.10 Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_5$ singularity, $\delta_P(S) = \frac{9}{7}$ 206
9.11 Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_6$ singularity 208
9.12 Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_6$ singularity, $\delta_P(S) = \frac{9}{8}$ 208
9.13 Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_6$ singularity, $\delta_P(S) = \frac{36}{29}$ 209
9.14 Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_7$ singularity (reducible ramification divisor) 211
9.15 Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_7$ singularity, $\delta_P(S) = 1$ 211
9.16 Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_7$ singularity, $\delta_P(S) = \frac{12}{11}$ 212
9.17 Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_7$ singularity, $\delta_P(S) = \frac{6}{5}$ 213
9.18 Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_7$ singularity (irreducible ramification divisor) 215
9.19 Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_7$ singularity, $\delta_P(S) = \frac{18}{17}$ (1) 215
9.20 Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_7$ singularity, $\delta_P(S) = \frac{18}{17}$ (2) 216
9.21 Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_8$ singularity 219
9.22 Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_8$ singularity, $\delta_P(S) = 1$ 219
9.23 Dual graph: $(-K_S)^2 = 1$, $\mathbb{D}_4$ singularity 222
9.24 Dual graph: $(-K_S)^2 = 1$, $\mathbb{D}_5$ singularity 223
9.25 Dual graph: $(-K_S)^2 = 1$, $\mathbb{D}_5$ singularity, $\delta_P(S) = \frac{4}{3}$ 224
9.26 Dual graph: $(-K_S)^2 = 1$, $\mathbb{D}_6$ singularity 226
9.27 Dual graph: $(-K_S)^2 = 1$, $\mathbb{D}_6$ singularity, $\delta_P(S) = \frac{6}{5}$ 227
9.28 Dual graph: $(-K_S)^2 = 1$, $\mathbb{D}_7$ singularity 229
9.29 Dual graph: $(-K_S)^2 = 1$, $\mathbb{D}_8$ singularity 232
9.30 Dual graph: $(-K_S)^2 = 1$, $\mathbb{E}_6$ singularity 236
9.31 Dual graph: $(-K_S)^2 = 1$, $\mathbb{E}_6$ singularity, $\delta_P(S) = \frac{3}{5}$ 237
9.32 Dual graph: $(-K_S)^2 = 1$, $\mathbb{E}_7$ singularity 239
9.33 Dual graph: $(-K_S)^2 = 1$, $\mathbb{E}_7$ singularity, $\delta_P(S) = \frac{6}{5}$ 242
9.34 Dual graph: $(-K_S)^2 = 1$, $\mathbb{E}_8$ singularity 243

12.1 Polarized δ-invariants: $\mathbb{A}_1$ singularity 290
12.2 Polarized δ-invariants: $2\mathbb{A}_1$ singularities 300
12.3 Polarized δ-invariants: $\mathbb{A}_2$ singularity 312

Tables

1.1 δ-invariants of Du Val del Pezzo surfaces of degrees 8, 7, 6, 5, 4, 3, 2 5
1.2 δ-invariants of Du Val del Pezzo surfaces of degree 1 6

2.1 δ-invariants of smooth del Pezzo surfaces 12
2.2 δ-invariants of Du Val del Pezzo surfaces of degrees 8, 7, 6, 5, 4, 3, 2 13
2.3 δ-invariants of Du Val del Pezzo surfaces of degree 1 14

3.1 δ-invariants of Du Val del Pezzo surfaces of degrees 7 and 8 17

4.1 δ-invariants of Du Val del Pezzo surfaces of degree 6 21
4.2 Local δ-invariants: $(-K_S)^2 = 6$ and $\mathbb{A}_1$ singularity, 3 lines 31
4.3 Local δ-invariants: $(-K_S)^2 = 6$ and $\mathbb{A}_1$ singularity, 4 lines 31
4.4 Local δ-invariants: $(-K_S)^2 = 6$ and $2\mathbb{A}_1$ singularities 31
4.5 Local δ-invariants: $(-K_S)^2 = 6$ and $\mathbb{A}_2$ singularity 31
4.6 Local δ-invariants: $(-K_S)^2 = 6$ and $\mathbb{A}_2\mathbb{A}_1$ singularities 31

5.1 δ-invariants of Du Val del Pezzo surfaces of degree 5 33
5.2 Local δ-invariants: $(-K_S)^2 = 5$ and $\mathbb{A}_1$ singularity 50
5.3 Local δ-invariants: $(-K_S)^2 = 5$ and $2\mathbb{A}_1$ singularities 50
5.4 Local δ-invariants: $(-K_S)^2 = 5$ and $\mathbb{A}_2$ singularity 50
5.5 Local δ-invariants: $(-K_S)^2 = 5$ and $\mathbb{A}_2\mathbb{A}_1$ singularities 50
5.6 Local δ-invariants: $(-K_S)^2 = 5$ and $\mathbb{A}_3$ singularity 50
5.7 Local δ-invariants: $(-K_S)^2 = 5$ and $\mathbb{A}_4$ singularity 51

6.2 δ-invariants of Du Val del Pezzo surfaces of degree 4 52
6.4 $(-K_S)^2 = 4$, a general point on a (-1)-curve 53
6.5 $(-K_S)^2 = 4$, the intersection of two (-1)-curves 54
6.6 Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{A}_1$ singularity 77
6.7 Local δ-invariants: $(-K_S)^2 = 4$ and $2\mathbb{A}_1$ singularities (9 lines) 77
6.8 Local δ-invariants: $(-K_S)^2 = 4$ and $2\mathbb{A}_1$ singularities (8 lines) 77
6.9 Local δ-invariants: $(-K_S)^2 = 4$ and $3\mathbb{A}_1$ singularities 77
6.10 Local δ-invariants: $(-K_S)^2 = 4$ and $4\mathbb{A}_1$ singularities 78
6.11 Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{A}_2$ singularity 78
6.12 Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{A}_2\mathbb{A}_1$ singularities 78
6.13 Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{A}_2 2\mathbb{A}_1$ singularities 78
6.14 Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{A}_3$ singularity (5 lines) 78
6.15 Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{A}_3$ singularity (4 lines) 78
6.16 Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{A}_3\mathbb{A}_1$ singularities 79
6.17 Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{A}_3 2\mathbb{A}_1$ singularities 79

6.18 Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{A}_4$ singularity . . . 79
6.19 Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{D}_4$ singularity . . . 79
6.20 Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{D}_5$ singularity . . . 79

7.2 δ-invariants of Du Val del Pezzo surfaces of degree 3 . . . 82
7.4 Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{A}_1$ singularity . . . 116
7.5 Local δ-invariants: $(-K_S)^2 = 3$ and $2\mathbb{A}_1$ singularities . . . 116
7.6 Local δ-invariants: $(-K_S)^2 = 3$ and $3\mathbb{A}_1$ singularities . . . 117
7.7 Local δ-invariants: $(-K_S)^2 = 3$ and $4\mathbb{A}_1$ singularities . . . 117
7.8 Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{A}_2$ singularity . . . 117
7.9 Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{A}_2\mathbb{A}_1$ singularities . . . 117
7.10 Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{A}_2 2\mathbb{A}_1$ singularities . . . 117
7.11 Local δ-invariants: $(-K_S)^2 = 3$ and $2\mathbb{A}_2$ singularities . . . 117
7.12 Local δ-invariants: $(-K_S)^2 = 3$ and $2\mathbb{A}_2\mathbb{A}_1$ singularities . . . 118
7.13 Local δ-invariants: $(-K_S)^2 = 3$ and $3\mathbb{A}_2$ singularities . . . 118
7.14 Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{A}_3$ singularity . . . 118
7.15 Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{A}_3\mathbb{A}_1$ singularities . . . 118
7.16 Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{A}_3 2\mathbb{A}_1$ singularities . . . 118
7.17 Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{A}_4$ singularity . . . 118
7.18 Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{A}_4\mathbb{A}_1$ singularities . . . 119
7.19 Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{A}_5$ singularity . . . 119
7.20 Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{A}_5\mathbb{A}_1$ singularities . . . 119
7.21 Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{D}_4$ singularity . . . 119
7.22 Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{D}_5$ singularity . . . 119
7.23 Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{E}_6$ singularity . . . 119

8.2 δ-invariants of Du Val del Pezzo surfaces of degree 2 . . . 123
8.4 Local δ-invariants: $(-K_S)^2 = 2$ and $2\mathbb{A}_1$ singularities . . . 177
8.5 Local δ-invariants: $(-K_S)^2 = 2$ and $3\mathbb{A}_1$ singularities (26 lines) . . . 177
8.6 Local δ-invariants: $(-K_S)^2 = 2$ and $3\mathbb{A}_1$ singularities (25 lines) . . . 177
8.7 Local δ-invariants: $(-K_S)^2 = 2$ and $4\mathbb{A}_1$ singularities (20 lines) . . . 177
8.8 Local δ-invariants: $(-K_S)^2 = 2$ and $4\mathbb{A}_1$ singularities (19 lines) . . . 177
8.9 Local δ-invariants: $(-K_S)^2 = 2$ and $5\mathbb{A}_1$ singularities . . . 177
8.10 Local δ-invariants: $(-K_S)^2 = 2$ and $6\mathbb{A}_1$ singularities . . . 178
8.11 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_2$ singularity . . . 178
8.12 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_2\mathbb{A}_1$ singularities . . . 178
8.13 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_2 2\mathbb{A}_1$ singularities . . . 178
8.14 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_2 3\mathbb{A}_1$ singularities . . . 178
8.15 Local δ-invariants: $(-K_S)^2 = 2$ and $2\mathbb{A}_2$ singularities . . . 179
8.16 Local δ-invariants: $(-K_S)^2 = 2$ and $2\mathbb{A}_2\mathbb{A}_1$ singularities . . . 179

8.17 Local δ-invariants: $(-K_S)^2 = 2$ and $3\mathbb{A}_2$ singularities 179
8.18 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_3$ singularity 179
8.19 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_3\mathbb{A}_1$ singularities (16 lines) 179
8.20 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_3\mathbb{A}_1$ singularities (15 lines) 180
8.21 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_3 2\mathbb{A}_1$ singularities (12 lines) 180
8.22 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_3 2\mathbb{A}_1$ singularities (11 lines) 180
8.23 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_3 3\mathbb{A}_1$ singularities 180
8.24 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_3\mathbb{A}_2$ singularities 180
8.25 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_3\mathbb{A}_2\mathbb{A}_1$ singularities 181
8.26 Local δ-invariants: $(-K_S)^2 = 2$ and $2\mathbb{A}_3$ singularities 181
8.27 Local δ-invariants: $(-K_S)^2 = 2$ and $2\mathbb{A}_3\mathbb{A}_1$ singularities 181
8.28 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_4$ singularity 181
8.29 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_4\mathbb{A}_1$ singularities 181
8.30 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_4\mathbb{A}_2$ singularities 182
8.31 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_5$ singularity (8 lines) 182
8.32 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_5$ singularity (7 lines) 182
8.33 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_5\mathbb{A}_1$ singularities (6 lines) 182
8.34 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_5\mathbb{A}_1$ singularities (5 lines) 182
8.35 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_5\mathbb{A}_2$ singularities 183
8.36 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_6$ singularity 183
8.37 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_7$ singularity 183
8.38 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{D}_4$ singularity 183
8.39 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{D}_4\mathbb{A}_1$ singularities 183
8.40 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{D}_4 2\mathbb{A}_1$ singularities 184
8.41 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{D}_4 3\mathbb{A}_1$ singularities 184
8.42 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{D}_5$ singularity 184
8.43 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{D}_5\mathbb{A}_1$ singularities 184
8.44 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{D}_6$ singularity 184
8.45 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{D}_6\mathbb{A}_1$ singularities 184
8.46 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{E}_6$ singularity 185
8.47 Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{E}_7$ singularity 185

9.1 δ-invariants of Du Val del Pezzo surfaces of degree 1 193

11.1 Intersections on S - del Pezzo 5 271
11.2 Zariski Decomposition of $(-K_X - uS)|_S - vZ$ for S - del Pezzo 5 272
11.3 Intersections on $E_L \cong \mathbb{P}^1 \times \mathbb{P}^1$ 273
11.4 Zariski Decomposition of $(-K_X - uE_L)|_{E_L} - vZ$ for $E_L \cong \mathbb{P}^1 \times \mathbb{P}^1$ (case 1) 274
11.5 Zariski Decomposition of $(-K_X - uE_L)|_{E_L} - vZ$ for $E_L \cong \mathbb{P}^1 \times \mathbb{P}^1$ (case 2) 275
11.6 Intersections on $\mathscr{S}$ - del Pezzo 6 (pic. 1) 276

11.7 Intersections on $\mathscr{S}$ - del Pezzo 6 (pic. 2) 276
11.8 Zariski Decomposition of $(-K_X - u\mathscr{S})|_{\mathscr{S}} - vZ$ for $\mathscr{S}$ - del Pezzo 6 (case 1) 277
11.9 Zariski Decomposition of $(-K_X - u\mathscr{S})|_{\mathscr{S}} - vZ$ for $\mathscr{S}$ - del Pezzo 6 (case 2) 278
11.10Zariski Decomposition of $(-K_X - u\mathscr{S})|_{\mathscr{S}} - vZ$ for $\mathscr{S}$ - del Pezzo 6 (case 3) 279

Chapter 1

Introduction

This thesis lies in the field of **Algebraic Geometry** — a branch of pure mathematics that blends algebra and geometry by using algebraic techniques to study geometric objects. At its core, Algebraic Geometry investigates **spaces with an algebraic structure**, known as **varieties**, which are defined as the **solutions of systems of polynomial equations**.

The recent advances in subclasses of algebraic geometry, such as birational geometry via the Minimal Model Program (MMP) have led to a deeper understanding of the structure of algebraic varieties. One of the major achievements of the MMP is a framework for classifying varieties into a few fundamental types, which can be thought of as the **building blocks** from which all other varieties are constructed. These three foundational classes are: **varieties of general type**, **Calabi–Yau varieties**, and **Fano varieties**. Each of these classes is characterized by the sign of the canonical divisor K_X: positive, trivial, or negative, respectively, which corresponds geometrically to the curvature properties of the variety.

In this thesis, we focus on **Fano varieties**, which are normal projective varieties with **klt (Kawamata log terminal)** singularities and an **ample anticanonical divisor** $-K_X$. The klt condition ensures that the singularities of the variety are mild and manageable within the framework of the MMP (the considered singularities are described more formally later in the thesis). The ampleness of the anticanonical divisor means that the variety X can be embedded into a projective space, and from a differential-geometric perspective, Fano varieties are the higher-dimensional analogues of positively curved spaces, like the sphere. These varieties are of particular interest because they sit at the heart of several key questions in algebraic geometry and complex differential geometry.

One of the key goals in complex geometry — a very important subfield of algebraic geometry, is to study when the above classes of varieties admit canonical metrics. A fundamental example of such metrics are **Kähler–Einstein metrics**. Aubin (1978) and Yau (1977), and later Yau (1978) in his seminal work, which won him the Fields medal in 1982, proved that general type and Calabi–Yau varieties always admit a unique Kähler–Einstein metric. However, it was clear that this is not true for Fano varieties since there were known examples of Fano varieties which did not admit a Kähler–Einstein metric. One of the earliest examples follows from the paper of Matsushima (1957) — a blow up of $\mathbb{P}^2$ at one point cannot have a Kähler–Einstein metric due to the nature of its automorphism group.

The explicit detection of Kähler–Einstein metrics on Fano varieties was a very difficult task, and it became an integral problem for algebraic geometers in the last quarter of the 20th century. Around the same time, a remarkable philosophical approach to tackle the existence of a Kähler–Einstein metric on Fano varieties was proposed. In 1993 Yau conjectured that an algebrogeometric notion of stability would be necessary to determine the existence of Kähler–Einstein metrics on Fano varieties. This was inspired by earlier work by Uhlenbeck and Yau (1986) and Donaldson (1985) predicting the existence of metrics on vector bundles using slope stability.

The notion of **K-stability** was introduced first in Tian (1997) and then reformulated in an algebraic way in Donaldson (2002); it was conjectured that the existence of these metrics would be equivalent to algebraic conditions on Fano varieties called K-stability conditions. This is known as the **Yau–Tian–Donaldson conjecture**. More formally, the Yau–Tian–Donaldson conjecture states that a Fano manifold admits a Kähler–Einstein metric if and only if it is K-polystable. This **conjecture was first proven by Chen-Donaldson-Sun** for smooth Fano varieties in *"Kähler–Einstein metrics on Fano manifolds I, II, III"* (see Chen, Donaldson, and Sun (2015)). Then, a similar result was proven for smoothable Fano varieties by Li, Wang, and Xu (2019) and Spotti, Sun, and Yao (2016) and the further expanded to a larger class of singular Fano varieties in the works of Li, Tian, and Wang (2021, 2022) and Liu, Xu, and Zhuang (2022).

There are several equivalent definitions of K-stability. In Tian (1997) it was originally defined in analytic terms using the α-invariant — an approach that was later shown to coincide with the global log canonical threshold introduced independently by Shokurov in birational geometry. This analytic perspective followed Yau's philosophical intuition (see Yau (1993)). Shortly thereafter, Donaldson (2002) provided an equivalent, purely algebro-geometric definition via the Donaldson–Futaki invariant, which is defined for any test configuration and therefore applies beyond the Fano case. To make the Donaldson–Futaki invariant more accessible for explicit computations, mathematicians sought to express it as an intersection number. Achieving this required working with compactification of the Donaldson's construction, and in this context, Odaka (2013) and Wang (2012) reformulated the Futaki invariant in terms of intersection theory.

DEFINITION. Let X be a Fano variety with $\dim(X) = n$ for $n \geq 2$ with Kawamata log terminal singularities. Set $L = -K_X$. A **test configuration** of a pair $(X;L)$ consists of the following data:

- a normal variety $\mathscr{X}$ with a $\mathbb{G}_m$ action,
- a flat $\mathbb{G}_m$-equivariant morphism $p\colon \mathscr{X} \to \mathbb{P}^1$; $\mathbb{G}_m$ acts on $\mathbb{P}^1$ by $(t,[x:y]) \mapsto [tx:y]$,
- a $\mathbb{G}_m$-invariant p-ample $\mathbb{Q}$-line bundle $\mathscr{L} \to \mathscr{X}$ and a $\mathbb{G}_m$-equivariant isomorphism

$$\big(\mathscr{X}\backslash p^{-1}(0), \mathscr{L}|_{\mathscr{X}\backslash p^{-1}(0)}\big) \cong \big(X \times (\mathbb{P}^1\backslash\{0\}), \mathrm{pr}_1^*(L)\big),$$

where pr_1 is the projection to the first factor, and $0 = [0:1]$, $\infty = [1:0]$.

The **Donaldson–Futaki invariant** of a test configuration $(\mathscr{X},\mathscr{L})$ is defined as:

$$\mathrm{DF}\big(\mathscr{X};\mathscr{L}\big) = \frac{1}{L^n}\Big(\mathscr{L}^n \cdot K_{\mathscr{X}/\mathbb{P}^1} + \frac{n}{n+1}\mathscr{L}^{n+1}\Big).$$

Denote the central fibre $p^{-1}(0)$ by $\mathscr{X}_0$, and the fibre at infinity $p^{-1}(\infty)$ by $\mathscr{X}_\infty$. The test configuration $(\mathscr{X},\mathscr{L})$ is called

- **trivial** if there is a $\mathbb{G}_m$-equivariant isomorphism

$$\big(\mathscr{X}\backslash\mathscr{X}_\infty, \mathscr{L}|_{\mathscr{X}\backslash\mathscr{X}_\infty}\big) \cong \big(X \times (\mathbb{P}^1\backslash\infty), \mathrm{pr}_1^*(L)\big),$$

- **product-type** if we have an isomorphism $\mathscr{X}\backslash\mathscr{X}_\infty \cong X \times (\mathbb{P}^1\backslash\infty)$.

The Fano variety X is called

- K**-semistable** if for every its test configuration $(\mathscr{X},\mathscr{L})$ one has $\mathrm{DF}(\mathscr{X};\mathscr{L}) \geqslant 0$,
- K**-stable** if for every its non-trivial test configuration $(\mathscr{X},\mathscr{L})$ one has $\mathrm{DF}(\mathscr{X};\mathscr{L}) > 0$,
- K**-polystable** if it is K-semistable and $\mathrm{DF}(\mathscr{X};\mathscr{L}) = 0$ iff $(\mathscr{X},\mathscr{L})$ is of the product type.

Note that: X is K-stable $\Longrightarrow$ X is K-polystable $\Longrightarrow$ X is K-semistable.

Later on, Fujita (2019) and Li (2017) were able to give a **valuative** definition of K-stability using ideas from birational geometry. In the work of Fujita and Odaka (2018) a new algebraic invariant called **the δ-invariant** was introduced. It is also known as **the stability threshold** due to the fact that its value reflects whether the variety in question is K-stable. In the works by Blum and Jonsson (2020); Fujita (2019); Fujita and Odaka (2018); Li (2017); Liu et al. (2022); Xu and Zhuang (2020) the following theorem was proven:

THEOREM. If X is a Fano variety, then

- $\delta(X) > 1 \iff X$ is K-stable,
- $\delta(X) \geqslant 1 \iff X$ is K-semistable.

One of the fundamental goals for algebraic geometers working on K-stability of Fano varieties has been to develop a **systematic approach for detecting K-stabilty** of Fano varieties and using that to obtain a comprehensive classification of K-stable Fano varieties. This goal has been largely achieved via the **method of Abban and Zhuang (2022)**, which allows to “reduce the dimension" of the problem. More precisely, the key idea is that subvarieties of codimension one provide the information about the initial variety.

Research on the detection of K-stability of Fano varieties has heavily focused on smooth Fano varieties of dimensions less than four. Unfortunately, K-stability of higher dimensional Fano varieties and singular Fano threefolds is generically unknown due to the complexity of calculating the invariants explicitly, and to the lack of a full classification of Fano varieties in dimensions higher than 3. Using the Abban–Zhuang method in many cases detecting K-stability can be reduced to computing an algebraic invariant on two dimensional Fano varieties — possibly singular del Pezzo surfaces.

In **Part I** of this thesis, we compute the δ-invariants of all del Pezzo surfaces with Du Val singularities. A **smooth del Pezzo surface** is a projective algebraic surface whose anticanonical divisor is ample. These surfaces are the two-dimensional Fano varieties and play a fundamental role in the classification of algebraic surfaces. The degree of a del Pezzo surface is defined as the self-intersection number of its canonical class, which ranges from 1 to 9. All smooth del Pezzo surfaces, except for $\mathbb{P}^1 \times \mathbb{P}^1$, can be obtained as a blowup of the the projective plane $\mathbb{P}^2$ at zero to eight points in general position. The degree decreases by one for each blow-up. In particular, $\mathbb{P}^2$ itself is the del Pezzo surface of degree 9.

The notion can be extended to allow mild singularities: a **singular del Pezzo surface** is a normal projective surface with only log terminal (for example, Du Val) singularities and an ample anticanonical divisor. These surfaces arise naturally in the MMP and in the compactification of moduli spaces. Du Val singularities are a subclass of canonical surface singularities; their description can be found in Reid (1987) or Dolgachev (2012).

In Part I of the thesis, we determine the stability thresholds of del Pezzo surfaces with Du Val singularities. For degrees 9 to 2, we consider all points on each of the Du Val del Pezzo surfaces to obtain the result. For degree 1, a slightly different approach was taken: we show that the stability threshold is achieved at the "worst" singularity and compute the δ-invariant at that point. Along the way, we provide a description of the Mori cones of Du Val del Pezzo surfaces of degree 2, which, to our knowledge, had not been previously presented. More precisely, in Part I we prove the following theorem:

MAIN THEOREM 1. Let X be a Du Val del Pezzo surface of degree d where $d \geq 2$. Then the δ-invariant of X is uniquely determined by the degree of X, the number of lines on X, and the type of singularities on X which is given in the following table:

K_X^2	# ls	Sing(X)	$\delta(X)$
8	0	$\mathbb{A}_1$	$\frac{3}{4}$
7	2	$\mathbb{A}_1$	$\frac{21}{31}$
6	3	$\mathbb{A}_1$	$\frac{3}{4}$
6	4	$\mathbb{A}_1$	$\frac{9}{11}$
6	2	$2\mathbb{A}_1$	$\frac{9}{14}$
6	2	$\mathbb{A}_2$	$\frac{3}{5}$
6	1	$\mathbb{A}_2+\mathbb{A}_1$	$\frac{1}{2}$
5	7	$\mathbb{A}_1$	$\frac{15}{17}$
5	5	$2\mathbb{A}_1$	$\frac{15}{19}$

K_X^2	# ls	Sing(X)	$\delta(X)$
5	4	$\mathbb{A}_2$	$\frac{5}{7}$
5	3	$\mathbb{A}_2+\mathbb{A}_1$	$\frac{15}{23}$
5	2	$\mathbb{A}_3$	$\frac{5}{9}$
5	1	$\mathbb{A}_4$	$\frac{3}{7}$
4	12	$\mathbb{A}_1$	1
4	9	$2\mathbb{A}_1$	1
4	8	$2\mathbb{A}_1$	1
4	6	$3\mathbb{A}_1$	1
4	4	$4\mathbb{A}_1$	1
4	8	$\mathbb{A}_2$	$\frac{6}{7}$

K_X^2	# ls	Sing(X)	$\delta(X)$
4	6	$\mathbb{A}_2+\mathbb{A}_1$	$\frac{6}{7}$
4	4	$\mathbb{A}_2+2\mathbb{A}_1$	$\frac{6}{7}$
4	5	$\mathbb{A}_3$	$\frac{2}{3}$
4	4	$\mathbb{A}_3$	$\frac{3}{4}$
4	3	$\mathbb{A}_3+\mathbb{A}_1$	$\frac{3}{4}$
4	2	$\mathbb{A}_3+2\mathbb{A}_1$	$\frac{3}{4}$
4	3	$\mathbb{A}_4$	$\frac{6}{11}$
4	2	$\mathbb{D}_4$	$\frac{1}{2}$
4	1	$\mathbb{D}_5$	$\frac{3}{8}$

K_X^2	# ls	Sing(X)	$\delta(X)$
3	21	$\mathbb{A}_1$	$\frac{6}{5}$
3	16	$2\mathbb{A}_1$	$\frac{6}{5}$
3	12	$3\mathbb{A}_1$	$\frac{6}{5}$
3	9	$4\mathbb{A}_1$	$\frac{6}{5}$
3	15	$\mathbb{A}_2$	1
3	11	$\mathbb{A}_2+\mathbb{A}_1$	1
3	8	$\mathbb{A}_2+2\mathbb{A}_1$	1

K_X^2	# ls	Sing(X)	$\delta(X)$
3	7	$2\mathbb{A}_2$	1
3	5	$2\mathbb{A}_2+\mathbb{A}_1$	1
3	3	$3\mathbb{A}_2$	1
3	10	$\mathbb{A}_3$	$\frac{9}{11}$
3	7	$\mathbb{A}_3+\mathbb{A}_1$	$\frac{9}{11}$
3	5	$\mathbb{A}_3+2\mathbb{A}_1$	$\frac{9}{11}$

K_X^2	# ls	Sing(X)	$\delta(X)$
3	6	$\mathbb{A}_4$	$\frac{9}{13}$
3	4	$\mathbb{A}_4+\mathbb{A}_1$	$\frac{9}{13}$
3	3	$\mathbb{A}_5$	$\frac{3}{5}$
3	2	$\mathbb{A}_5+\mathbb{A}_1$	$\frac{3}{5}$
3	6	$\mathbb{D}_4$	$\frac{3}{5}$
3	3	$\mathbb{D}_5$	$\frac{9}{19}$
3	1	$\mathbb{E}_6$	$\frac{1}{3}$

K_X^2	# ls	Sing(X)	$\delta(X)$
2	44	$\mathbb{A}_1$	$\frac{3}{2}$
2	34	$2\mathbb{A}_1$	$\frac{3}{2}$
2	26	$3\mathbb{A}_1$	$\frac{3}{2}$
2	25	$3\mathbb{A}_1$	$\frac{3}{2}$
2	20	$4\mathbb{A}_1$	$\frac{3}{2}$
2	19	$4\mathbb{A}_1$	$\frac{3}{2}$
2	14	$5\mathbb{A}_1$	$\frac{3}{2}$
2	10	$6\mathbb{A}_1$	$\frac{3}{2}$
2	31	$\mathbb{A}_2$	$\frac{6}{5}$
2	20	$\mathbb{A}_2+\mathbb{A}_1$	$\frac{6}{5}$
2	18	$\mathbb{A}_2+2\mathbb{A}_1$	$\frac{6}{5}$
2	13	$\mathbb{A}_2+3\mathbb{A}_1$	$\frac{6}{5}$
2	16	$2\mathbb{A}_2$	$\frac{6}{5}$
2	12	$2\mathbb{A}_2+\mathbb{A}_1$	$\frac{6}{5}$
2	8	$3\mathbb{A}_2$	$\frac{6}{5}$

K_X^2	# ls	Sing(X)	$\delta(X)$
2	22	$\mathbb{A}_3$	1
2	16	$\mathbb{A}_3+\mathbb{A}_1$	1
2	15	$\mathbb{A}_3+\mathbb{A}_1$	1
2	12	$\mathbb{A}_3+2\mathbb{A}_1$	1
2	11	$\mathbb{A}_3+2\mathbb{A}_1$	1
2	8	$\mathbb{A}_3+3\mathbb{A}_1$	1
2	10	$\mathbb{A}_3+\mathbb{A}_2$	1
2	7	$\mathbb{A}_3+\mathbb{A}_2+\mathbb{A}_1$	1
2	6	$2\mathbb{A}_3$	1
2	4	$2\mathbb{A}_3+\mathbb{A}_1$	1
2	14	$\mathbb{A}_4$	$\frac{12}{13}$
2	10	$\mathbb{A}_4+\mathbb{A}_1$	$\frac{12}{13}$
2	6	$\mathbb{A}_4+\mathbb{A}_2$	$\frac{12}{13}$
2	8	$\mathbb{A}_5$	$\frac{6}{7}$
2	7	$\mathbb{A}_5$	$\frac{3}{4}$

K_X^2	# ls	Sing(X)	$\delta(X)$
2	6	$\mathbb{A}_5+\mathbb{A}_1$	$\frac{6}{7}$
2	5	$\mathbb{A}_5+\mathbb{A}_1$	$\frac{3}{4}$
2	3	$\mathbb{A}_5+\mathbb{A}_2$	$\frac{3}{4}$
2	4	$\mathbb{A}_6$	$\frac{4}{5}$
2	2	$\mathbb{A}_7$	$\frac{3}{4}$
2	14	$\mathbb{D}_4$	$\frac{3}{4}$
2	9	$\mathbb{D}_4+\mathbb{A}_1$	$\frac{3}{4}$
2	6	$\mathbb{D}_4+2\mathbb{A}_1$	$\frac{3}{4}$
2	4	$\mathbb{D}_4+3\mathbb{A}_1$	$\frac{3}{4}$
2	8	$\mathbb{D}_5$	$\frac{3}{5}$
2	5	$\mathbb{D}_5+\mathbb{A}_1$	$\frac{3}{5}$
2	3	$\mathbb{D}_6$	$\frac{1}{2}$
2	2	$\mathbb{D}_6+\mathbb{A}_1$	$\frac{1}{2}$
2	4	$\mathbb{E}_6$	$\frac{3}{7}$
2	1	$\mathbb{E}_7$	$\frac{3}{10}$

Table 1.1: δ-invariants of Du Val del Pezzo surfaces of degrees 8, 7, 6, 5, 4, 3, 2

MAIN THEOREM 2. Let X be a Du Val del Pezzo surface of degree 1. Then, X can be realized as the double cover $X \xrightarrow{2:1} \mathbb{P}(1,1,2)$, which is ramified along a sextic curve $R \in \mathbb{P}(1,1,2)$. Then, the δ-invariant of X is uniquely determined by the by the type of singularities on X and unique elements of $|-K_X|$ containing each of singular points which is given in the following table:

Type of singularity	$\delta(X)$
$\mathbb{A}_1$, $2\mathbb{A}_1$, $3\mathbb{A}_1$, $4\mathbb{A}_1$, $5\mathbb{A}_1$, $6\mathbb{A}_1$, $7\mathbb{A}_1$, $8\mathbb{A}_1$ all elements of $|-K_X|$ containing singular points are nodal	2
$\mathbb{A}_1$, $2\mathbb{A}_1$, $3\mathbb{A}_1$, $4\mathbb{A}_1$, $5\mathbb{A}_1$, $6\mathbb{A}_1$, $7\mathbb{A}_1$, $8\mathbb{A}_1$ some elements of $|-K_X|$ containing singular points are cuspidal	$\frac{9}{5}$
$\mathbb{A}_2$, $\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_2+2\mathbb{A}_1$, $\mathbb{A}_2+3\mathbb{A}_1$, $\mathbb{A}_2+4\mathbb{A}_1$, $2\mathbb{A}_2$, $2\mathbb{A}_2+\mathbb{A}_1$, $2\mathbb{A}_2+2\mathbb{A}_1$, $3\mathbb{A}_2$, $3\mathbb{A}_2+\mathbb{A}_1$, $4\mathbb{A}_2$ all elements of $|-K_X|$ containing $\mathbb{A}_2$ singular points are nodal	$\frac{12}{7}$
$\mathbb{A}_2$, $\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_2+2\mathbb{A}_1$, $\mathbb{A}_2+3\mathbb{A}_1$, $\mathbb{A}_2+4\mathbb{A}_1$, $2\mathbb{A}_2$, $2\mathbb{A}_2+\mathbb{A}_1$, $2\mathbb{A}_2+2\mathbb{A}_1$, $3\mathbb{A}_2$, $3\mathbb{A}_2+\mathbb{A}_1$, $4\mathbb{A}_2$ some elements of $|-K_X|$ containing $\mathbb{A}_2$ singular points are cuspidal	$\frac{3}{2}$
$\mathbb{A}_3$, $\mathbb{A}_3+\mathbb{A}_1$, $\mathbb{A}_3+2\mathbb{A}_1$, $\mathbb{A}_3+3\mathbb{A}_1$, $\mathbb{A}_3+4\mathbb{A}_1$, $\mathbb{A}_3+\mathbb{A}_2$, $\mathbb{A}_3+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_3+\mathbb{A}_2+2\mathbb{A}_1$, $2\mathbb{A}_3$, $2\mathbb{A}_3+\mathbb{A}_1$, $2\mathbb{A}_3+2\mathbb{A}_1$	$\frac{3}{2}$
$\mathbb{A}_4$, $\mathbb{A}_4+\mathbb{A}_1$, $\mathbb{A}_4+2\mathbb{A}_1$, $\mathbb{A}_4+\mathbb{A}_2$, $\mathbb{A}_4+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_4+\mathbb{A}_3$, $2\mathbb{A}_4$	$\frac{4}{3}$
$\mathbb{A}_5$, $\mathbb{A}_5+\mathbb{A}_1$, $\mathbb{A}_5+2\mathbb{A}_1$, $\mathbb{A}_5+\mathbb{A}_2$, $\mathbb{A}_5+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_5+\mathbb{A}_3$	$\frac{6}{5}$
$\mathbb{A}_6$, $\mathbb{A}_6+\mathbb{A}_1$	$\frac{9}{8}$
$\mathbb{A}_7$, $\mathbb{A}_7+\mathbb{A}_1$ and R irreducible	$\frac{18}{17}$
$\mathbb{A}_7$, $\mathbb{A}_7+\mathbb{A}_1$ and R reducible	1
$\mathbb{A}_8$, $\mathbb{D}_4$, $\mathbb{D}_4+\mathbb{A}_1$, $\mathbb{D}_4+2\mathbb{A}_1$, $\mathbb{D}_4+3\mathbb{A}_1$, $\mathbb{D}_4+4\mathbb{A}_1$,$\mathbb{D}_4+\mathbb{A}_2$, $\mathbb{D}_4+\mathbb{A}_3$, $2\mathbb{D}_4$	1
$\mathbb{D}_5$, $\mathbb{D}_5+\mathbb{A}_1$, $\mathbb{D}_5+2\mathbb{A}_1$, $\mathbb{D}_5+\mathbb{A}_2$, $\mathbb{D}_5+\mathbb{A}_3$	$\frac{6}{7}$
$\mathbb{D}_6$, $\mathbb{D}_6+\mathbb{A}_1$, $\mathbb{D}_6+2\mathbb{A}_1$	$\frac{3}{4}$
$\mathbb{D}_7$	$\frac{2}{3}$
$\mathbb{D}_8$, $\mathbb{E}_6$, $\mathbb{E}_6+\mathbb{A}_1$, $\mathbb{E}_6+\mathbb{A}_2$	$\frac{3}{5}$
$\mathbb{E}_7$, $\mathbb{E}_7+\mathbb{A}_1$	$\frac{3}{7}$
$\mathbb{E}_8$	$\frac{3}{11}$

Table 1.2: δ-invariants of Du Val del Pezzo surfaces of degree 1

This thesis presents the first complete computation of the δ-invariants for all del Pezzo surfaces with Du Val singularities. While estimates of these invariants were previously known, for instance, through the work of Odaka, Spotti, and Sun (2016) and Mabuchi and Mukai (1993) on the moduli of del Pezzo surfaces — these results typically provided only qualitative information, such as whether $\delta > 1$ or $\delta < 1$, based on the type of singularities. In contrast, the results presented here yield explicit values of δ in every case. This comprehensive computation not only completes the picture for del Pezzo surfaces but also plays a crucial role in determining

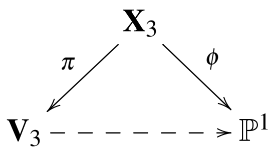
K-stability in higher-dimensional Fano varieties as we see in further corollaries and the proof of Main Theorem 4. In several instances, these higher-dimensional examples had previously unknown stability behaviour, which could not have been resolved without the precise δ-invariants computed in this work.

In **Part II** the existence of a Kähler–Einstein metric on Fano threefolds is discussed. As was discussed earlier, a Fano variety admits a Kähler–Einstein metric if and only if it is K-polystable. For two-dimensional Fano varieties (del Pezzo surfaces) Tian and Yau proved that a smooth del Pezzo surface is K-polystable if and only if it is not a blow up of $\mathbb{P}^2$ in one or two points (see Tian (1990); Tian and Yau (1987)). For three-dimensional Fano varieties the situation is more challenging. Smooth Fano threefolds over the field $\mathbb{C}$ have been classified be Iskovskikh (1997, 1998); Mori and Mukai (1981, 2003) into 105 families and the detailed description of these families can as well be found in Araujo et al. (2023). The results in Part I lead to finding the examples of Fano threefolds which admit a Kähler–Einstein metric:

Let $\mathbf{V}_3$ be a Fano threefold with canonical Gorenstein singularities such that $-K_{\mathbf{V}} \sim 2H$ for some $H \in \mathrm{Pic}(\mathbf{V}_3)$ with $H^3 = 3$, i.e $\mathbf{V}_3$ is a Fano threefold in Family №**1.13**.

COROLLARY 1. *If for any point Q on $\mathbf{V}_3$ there exists an element $X \in |H|$ such that $Q \in X$ and X is smooth then $\mathbf{V}_3$ is K-stable.*

Let S_1 and S_2 be two distinct surfaces in the linear system $|H|$, and let $\mathscr{C} = S_1 \cap S_2$. Suppose that the curve $\mathscr{C}$ is smooth. Then $\mathscr{C}$ is an elliptic curve by the adjunction formula. Let $\pi : \mathbf{X}_3 \to \mathbf{V}_3$ be the blow up of the curve $\mathscr{C}$, and let E be the π-exceptional surface. $\mathbf{X}_3$ is a Fano threefold in Family №**2.5**. We have the following commutative diagram:



Where $\mathbf{V}_3 \dashrightarrow \mathbb{P}^1$ is the rational map given by the pencil that is generated by S_1 and S_2, and ϕ is a fibration into del Pezzo surfaces of degree 3.

COROLLARY 2. *If every fiber X of ϕ at most $\mathbb{A}_2$ singularities, then $\mathbf{X}_3$ is K-stable.*

Let $\psi : \mathbf{V}_2 \to \mathbb{P}^3$ be the double cover branched along the reduced possibly reducible quartic surface R. Set $H = \psi^*(\mathscr{O}_{\mathbb{P}^3}(1))$. Then $\mathbf{V}_2$ is a del Pezzo threefold of degree 2 and a Fano threefold in Family №**1.12**. One can show that a general surface in $|H|$ is a smooth del Pezzo surface of degree 2.

COROLLARY 3. *If R has $\mathbb{A}_n$-singulalrities, then $\mathbf{V}_2$ is K-stable.*

Let S_1 and S_2 be two distinct surfaces in the linear system $|H|$, and let $\mathscr{C} = S_1 \cap S_2$. Suppose that the curve $\mathscr{C}$ is smooth. Then $\mathscr{C}$ is an elliptic curve by the adjunction formula. Let $\pi : \mathbf{X}_2 \to \mathbf{V}_2$ be the blow up of the curve $\mathscr{C}$, and let E be the π-exceptional surface. $\mathbf{X}_2$ is a Fano threefold in Family №**2.3**. We have the following commutative diagram:

$$\begin{array}{ccc} & \mathbf{X}_2 & \\ {}^{\pi}\swarrow & & \searrow^{\phi} \\ \mathbf{V}_2 & \dashrightarrow & \mathbb{P}^1 \end{array}$$

Where $\mathbf{V}_2 \dashrightarrow \mathbb{P}^1$ is the rational map given by the pencil that is generated by S_1 and S_2, and ϕ is a fibration into del Pezzo surfaces of degree 2.

COROLLARY 4. *If every fiber X of ϕ at most $\mathbb{A}_3$ singularities, then $\mathbf{X}_2$ is K-stable.*

Let $\mathbf{V}_1$ be a Fano threefold with canonical Gorenstein singularities such that $-K_{\mathbf{V}_1} \sim 2H$ for some $H \in \mathrm{Pic}(\mathbf{V}_1)$ with $H^3 = 1$, i.e $\mathbf{V}_1$ is a Fano threefold in Family №**1.11**. A general element in $|H|$ is a Du Val del Pezzo surface of degree 1 and if $\mathbf{V}_1$ has isolated singularities then a general surface in $|H|$ is a smooth.

COROLLARY 5. *If for any point Q on $\mathbf{V}_1$ there exists an element $X \in |H|$ such that $Q \in X$ and X has at most $\mathbb{A}_2$ singularities then $\mathbf{V}_1$ is K-stable.*

Let S_1 and S_2 be two distinct surfaces in the linear system $|H|$, and let $\mathscr{C} = S_1 \cap S_2$. Suppose that the curve $\mathscr{C}$ is smooth. Then $\mathscr{C}$ is an elliptic curve by the adjunction formula. Let $\pi : \mathbf{X}_1 \to \mathbf{V}_1$ be the blow up of the curve $\mathscr{C}$, and let E be the π-exceptional surface. $\mathbf{X}_1$ is a Fano threefold in Family №**2.1**. We have the following commutative diagram:

$$\begin{array}{ccc} & \mathbf{X}_1 & \\ {}^{\pi}\swarrow & & \searrow^{\phi} \\ \mathbf{V}_1 & \dashrightarrow & \mathbb{P}^1 \end{array}$$

Where $\mathbf{V}_1 \dashrightarrow \mathbb{P}^1$ is the rational map given by the pencil that is generated by S_1 and S_2, and ϕ is a fibration into del Pezzo surfaces of degree 1.

COROLLARY 6. *If every fiber X of ϕ at most $\mathbb{D}_4$ singularities, then $\mathbf{X}_1$ is K-stable.*

Let $R_{(2,4)}$ be a surface with isolated singularities of degree $(2,4)$ in $\mathbb{P}^1 \times \mathbb{P}^2$, let $\pi : \mathbf{X} \to \mathbb{P}^1 \times \mathbb{P}^2$ be a double cover ramified over the surface $R_{(2,4)}$, i.e $\mathbf{X}$ is a Fano threefold in Family №**2.2**. Let $pr_1 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ be the projection on the first factor. Set $p_1 = pr_1 \circ \pi$. Then p_1 is a fibration into del Pezzo surfaces of degree 2.

COROLLARY 7. *If every fiber X of p_1 has at most $\mathbb{A}_3$ singularities, then $\mathbf{X}$ is K-stable.*

REMARK Every smooth element in Family №1.13. is known to be K-stable by Araujo et al. (2023). Every smooth threefold in Family 2.5 such that there is no fiber of p_1 which contains $\mathbb{D}_5$ or $\mathbb{E}_6$ singularity in this family is known to be K-stable by Cheltsov, Denisova, and Fujita (2024). Every smooth element in Family №2.2. is known to be K-stable by Cheltsov et al. (2024). Every smooth element in Family №1.12. is known to be K-stable by Araujo et al. (2023) and Dervan (2016). Singular Del Pezzo Threefolds of degree 2 were studied in Ascher, DeVleming, and Liu (2023). It follows from Ascher et al. (2023); Shah (1981) that the threefold in Family №1.12 is K-polystable if and only if the quartic surface R is GIT-polystable with respect to natural action $PGL(4)$ except for those of the form $(x_0x_2+x_1^2+x_3^2)^2+a\cdot x_3^4$ for $a\in\mathbb{C}$. The corollary given by the Main Theorem is slightly weaker. Every smooth threefold in Family №2.3. is known to be K-stable by Cheltsov et al. (2024). Every smooth element in Family №1.11. is K-stable by Araujo et al. (2023). Every smooth Fano threefold in Family №2.1. is K-stable by Cheltsov et al. (2024).

These findings provide a wealth of new results on the K-stability of Fano varieties, many of which were previously unknown. Notably, they apply not only to smooth Fano threefolds but also to singular Fano varieties, which are significantly more difficult to study. Establishing K-stability in the singular setting typically requires much deeper analysis and is less understood than in the smooth case. The results presented here therefore mark a substantial advancement in the field, extending the landscape of known K-stable Fano varieties and opening the door to further developments in higher-dimensional birational geometry and moduli theory.

In the rest of the thesis, we consider different deformation families, and for one of them we solve the Calabi problem posed in Araujo et al. (2023) in its entirety:

Calabi Problem. *Find all K-polystable smooth Fano threefolds in each family.*

More precisely, we solve the Calabi problem completely for Family №**3.12**. A member of Family №3.12 X can be described as the blowup $\pi: X\to\mathbb{P}^3$ of $\mathbb{P}^3$ at a twisted cubic C and line L that is disjoint from C. We prove that:

MAIN THEOREM 3. All the smooth threefolds except one in Family №3.12 are K-polystable.

Hence, all smooth Fano threefolds in Family №3.12 except one described in (Araujo et al., 2023, §7.7) admit a Kähler–Einstein metric.

Now we approach the Calabi Problem for Family №**3.5**. Let C be a smooth curve in $S=\mathbb{P}^1\times\mathbb{P}^1$ of degree $(5,1)$. We consider embedding $S\hookrightarrow\mathbb{P}^1\times\mathbb{P}^2$ given by

$$\big([u:v],[x:y]\big)\mapsto\big([u:v],[x^2:xy:y^2]\big),$$

and identify S and C with their images in $\mathbb{P}^1 \times \mathbb{P}^2$. Let $\pi\colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ be the blow up of the curve C. Then, X is a smooth Fano threefold in the deformation Family № 3.5 in the Mori–Mukai list. Let $\mathrm{pr}_1 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ be the projection to the first factor and $\phi_1 = \mathrm{pr}_1 \circ \pi$. Then ϕ_1 is a fibration into del Pezzo surfaces of degree four, and every singular fiber of this fibration has Du Val singular points of types $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$ or $\mathbb{A}_4$. We prove the result which is a crucial step in solving the Calabi Problem for Family №3.5:

MAIN THEOREM 4. *If every singular fiber of ϕ_1 has only singular points of type $\mathbb{A}_1$, then X is K-stable.*

Although we do not solve the Calabi problem completely for Family № 3.5, Main Theorem 4 is the closest and most complete result (in combination with (Araujo et al., 2023, section 5.14)) in solving the Calabi problem for this family using the Abban-Zhuang method.

The main results of this thesis have appeared in a series of papers, such as Denisova (2023-24, 2024a, 2024b)

PART I

δ-invariants of Du Val Del Pezzo surfaces

Chapter 2

δ-invariants of Du Val Del Pezzo surfaces

2.1 History and known results

It is known that a smooth Fano variety admits a Kähler–Einstein metric if and only if it is K-polystable. In the case of two-dimensional Fano varieties (del Pezzo surfaces), Tian and Yau showed that a smooth del Pezzo surface is K-polystable if and only if it is not the blow-up of $\mathbb{P}^2$ at one or two points (see Tian (1990); Tian and Yau (1987)). Significant progress has been made for smooth Fano threefolds. However, for Fano varieties in higher dimensions, many questions remain open. In several cases, the problem reduces to computing the δ-invariant of (possibly singular) del Pezzo surfaces (see Araujo et al. (2023); Cheltsov et al. (2024); Cheltsov, Fujita, Kishimoto, and Okada (2023), etc.). For smooth del Pezzo surfaces δ-invariants were computed in Araujo et al. (2023):

X	$\mathbb{P}^2$	$\mathbb{P}^1\times\mathbb{P}^1$	S_8	S_7	S_6	S_5	S_4	S_3^1	S_3^2	S_2^1	S_2^2	S_1^1	S_1^2
$\delta(X)$	1	1	$\frac{6}{7}$	$\frac{21}{25}$	1	$\frac{15}{13}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{17}$	$\frac{9}{5}$	$\frac{15}{8}$	$\frac{15}{7}$	$\frac{12}{5}$

Table 2.1: δ-invariants of smooth del Pezzo surfaces

where S_d is a blowup of $\mathbb{P}^2$ at $9-d$ points in general position; S_3^1 is S_3 with an Eckardt point, S_3^2 is S_3 without an Eckardt point; S_2^1 is S_2 such that the linear system $|-K_{S_2}|$ contains a tacnodal curve; S_2^2 is S_2 such that the linear system $|-K_{S_2}|$ does not contain a tacnodal curve; S_1^1 is S_1 such that the linear system $|-K_{S_1}|$ contains a cuspidal curve; S_1^2 is S_1 such that the linear system $|-K_{S_1}|$ does not contain a cuspidal curve. In this article, we compute δ-invariants of singular Du Val del Pezzo surfaces. We prove that:

MAIN THEOREM 1. Let X be a Du Val del Pezzo surface of degree d where $d\geq 2$. Then the δ-invariant of X is uniquely determined by the degree of X, the number of lines on X, and the type of singularities on X which is given in the following table:

K_X^2	# ls	$\mathrm{Sing}(X)$	$\delta(X)$
8	0	$\mathbb{A}_1$	$\frac{3}{4}$
7	2	$\mathbb{A}_1$	$\frac{21}{31}$
6	3	$\mathbb{A}_1$	$\frac{3}{4}$
6	4	$\mathbb{A}_1$	$\frac{9}{11}$
6	2	$2\mathbb{A}_1$	$\frac{9}{14}$
6	2	$\mathbb{A}_2$	$\frac{3}{5}$
6	1	$\mathbb{A}_2+\mathbb{A}_1$	$\frac{1}{2}$
5	7	$\mathbb{A}_1$	$\frac{15}{17}$
5	5	$2\mathbb{A}_1$	$\frac{15}{19}$

K_X^2	# ls	$\mathrm{Sing}(X)$	$\delta(X)$
5	4	$\mathbb{A}_2$	$\frac{5}{7}$
5	3	$\mathbb{A}_2+\mathbb{A}_1$	$\frac{15}{23}$
5	2	$\mathbb{A}_3$	$\frac{5}{9}$
5	1	$\mathbb{A}_4$	$\frac{3}{7}$
4	12	$\mathbb{A}_1$	1
4	9	$2\mathbb{A}_1$	1
4	8	$2\mathbb{A}_1$	1
4	6	$3\mathbb{A}_1$	1
4	4	$4\mathbb{A}_1$	1
4	8	$\mathbb{A}_2$	$\frac{6}{7}$

K_X^2	# ls	$\mathrm{Sing}(X)$	$\delta(X)$
4	6	$\mathbb{A}_2+\mathbb{A}_1$	$\frac{6}{7}$
4	4	$\mathbb{A}_2+2\mathbb{A}_1$	$\frac{6}{7}$
4	5	$\mathbb{A}_3$	$\frac{2}{3}$
4	4	$\mathbb{A}_3$	$\frac{3}{4}$
4	3	$\mathbb{A}_3+\mathbb{A}_1$	$\frac{3}{4}$
4	2	$\mathbb{A}_3+2\mathbb{A}_1$	$\frac{3}{4}$
4	3	$\mathbb{A}_4$	$\frac{6}{11}$
4	2	$\mathbb{D}_4$	$\frac{1}{2}$
4	1	$\mathbb{D}_5$	$\frac{3}{8}$

K_X^2	# ls	$\mathrm{Sing}(X)$	$\delta(X)$
3	21	$\mathbb{A}_1$	$\frac{6}{5}$
3	16	$2\mathbb{A}_1$	$\frac{6}{5}$
3	12	$3\mathbb{A}_1$	$\frac{6}{5}$
3	9	$4\mathbb{A}_1$	$\frac{6}{5}$
3	15	$\mathbb{A}_2$	1
3	11	$\mathbb{A}_2+\mathbb{A}_1$	1
3	8	$\mathbb{A}_2+2\mathbb{A}_1$	1

K_X^2	# ls	$\mathrm{Sing}(X)$	$\delta(X)$
3	7	$2\mathbb{A}_2$	1
3	5	$2\mathbb{A}_2+\mathbb{A}_1$	1
3	3	$3\mathbb{A}_2$	1
3	10	$\mathbb{A}_3$	$\frac{9}{11}$
3	7	$\mathbb{A}_3+\mathbb{A}_1$	$\frac{9}{11}$
3	5	$\mathbb{A}_3+2\mathbb{A}_1$	$\frac{9}{11}$

K_X^2	# ls	$\mathrm{Sing}(X)$	$\delta(X)$
3	6	$\mathbb{A}_4$	$\frac{9}{13}$
3	4	$\mathbb{A}_4+\mathbb{A}_1$	$\frac{9}{13}$
3	3	$\mathbb{A}_5$	$\frac{3}{5}$
3	2	$\mathbb{A}_5+\mathbb{A}_1$	$\frac{3}{5}$
3	6	$\mathbb{D}_4$	$\frac{3}{5}$
3	3	$\mathbb{D}_5$	$\frac{9}{19}$
3	1	$\mathbb{E}_6$	$\frac{1}{3}$

K_X^2	# ls	$\mathrm{Sing}(X)$	$\delta(X)$
2	44	$\mathbb{A}_1$	$\frac{3}{2}$
2	34	$2\mathbb{A}_1$	$\frac{3}{2}$
2	26	$3\mathbb{A}_1$	$\frac{3}{2}$
2	25	$3\mathbb{A}_1$	$\frac{3}{2}$
2	20	$4\mathbb{A}_1$	$\frac{3}{2}$
2	19	$4\mathbb{A}_1$	$\frac{3}{2}$
2	14	$5\mathbb{A}_1$	$\frac{3}{2}$
2	10	$6\mathbb{A}_1$	$\frac{3}{2}$
2	31	$\mathbb{A}_2$	$\frac{6}{5}$
2	20	$\mathbb{A}_2+\mathbb{A}_1$	$\frac{6}{5}$
2	18	$\mathbb{A}_2+2\mathbb{A}_1$	$\frac{6}{5}$
2	13	$\mathbb{A}_2+3\mathbb{A}_1$	$\frac{6}{5}$
2	16	$2\mathbb{A}_2$	$\frac{6}{5}$
2	12	$2\mathbb{A}_2+\mathbb{A}_1$	$\frac{6}{5}$
2	8	$3\mathbb{A}_2$	$\frac{6}{5}$

K_X^2	# ls	$\mathrm{Sing}(X)$	$\delta(X)$
2	22	$\mathbb{A}_3$	1
2	16	$\mathbb{A}_3+\mathbb{A}_1$	1
2	15	$\mathbb{A}_3+\mathbb{A}_1$	1
2	12	$\mathbb{A}_3+2\mathbb{A}_1$	1
2	11	$\mathbb{A}_3+2\mathbb{A}_1$	1
2	8	$\mathbb{A}_3+3\mathbb{A}_1$	1
2	10	$\mathbb{A}_3+\mathbb{A}_2$	1
2	7	$\mathbb{A}_3+\mathbb{A}_2+\mathbb{A}_1$	1
2	6	$2\mathbb{A}_3$	1
2	4	$2\mathbb{A}_3+\mathbb{A}_1$	1
2	14	$\mathbb{A}_4$	$\frac{12}{13}$
2	10	$\mathbb{A}_4+\mathbb{A}_1$	$\frac{12}{13}$
2	6	$\mathbb{A}_4+\mathbb{A}_2$	$\frac{12}{13}$
2	8	$\mathbb{A}_5$	$\frac{6}{7}$
2	7	$\mathbb{A}_5$	$\frac{3}{4}$

K_X^2	# ls	$\mathrm{Sing}(X)$	$\delta(X)$
2	6	$\mathbb{A}_5+\mathbb{A}_1$	$\frac{6}{7}$
2	5	$\mathbb{A}_5+\mathbb{A}_1$	$\frac{3}{4}$
2	3	$\mathbb{A}_5+\mathbb{A}_2$	$\frac{3}{4}$
2	4	$\mathbb{A}_6$	$\frac{4}{5}$
2	2	$\mathbb{A}_7$	$\frac{3}{4}$
2	14	$\mathbb{D}_4$	$\frac{3}{4}$
2	9	$\mathbb{D}_4+\mathbb{A}_1$	$\frac{3}{4}$
2	6	$\mathbb{D}_4+2\mathbb{A}_1$	$\frac{3}{4}$
2	4	$\mathbb{D}_4+3\mathbb{A}_1$	$\frac{3}{4}$
2	8	$\mathbb{D}_5$	$\frac{3}{5}$
2	5	$\mathbb{D}_5+\mathbb{A}_1$	$\frac{3}{5}$
2	3	$\mathbb{D}_6$	$\frac{1}{2}$
2	2	$\mathbb{D}_6+\mathbb{A}_1$	$\frac{1}{2}$
2	4	$\mathbb{E}_6$	$\frac{3}{7}$
2	1	$\mathbb{E}_7$	$\frac{3}{10}$

Table 2.2: δ-invariants of Du Val del Pezzo surfaces of degrees 8, 7, 6, 5, 4, 3, 2

MAIN THEOREM 2. Let X be a Du Val del Pezzo surface of degree 1. Then X can be realized as the double cover $X \xrightarrow{2:1} \mathbb{P}(1,1,2)$, which is ramified along a sextic curve $R \in \mathbb{P}(1,1,2)$. Then the δ-invariant of X is uniquely determined by the by the type of singularities on X and unique elements of $|-K_X|$ containing each of singular points which is given in the following table:

Type of singularity	$\delta(X)$
$\mathbb{A}_1$, $2\mathbb{A}_1$, $3\mathbb{A}_1$, $4\mathbb{A}_1$, $5\mathbb{A}_1$, $6\mathbb{A}_1$, $7\mathbb{A}_1$, $8\mathbb{A}_1$ all elements of $|-K_X|$ containing singular points are nodal	2
$\mathbb{A}_1$, $2\mathbb{A}_1$, $3\mathbb{A}_1$, $4\mathbb{A}_1$, $5\mathbb{A}_1$, $6\mathbb{A}_1$, $7\mathbb{A}_1$, $8\mathbb{A}_1$ some elements of $|-K_X|$ containing singular points are cuspidal	$\frac{9}{5}$
$\mathbb{A}_2$, $\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_2+2\mathbb{A}_1$, $\mathbb{A}_2+3\mathbb{A}_1$, $\mathbb{A}_2+4\mathbb{A}_1$, $2\mathbb{A}_2$, $2\mathbb{A}_2+\mathbb{A}_1$, $2\mathbb{A}_2+2\mathbb{A}_1$, $3\mathbb{A}_2$, $3\mathbb{A}_2+\mathbb{A}_1$, $4\mathbb{A}_2$ all elements of $|-K_X|$ containing $\mathbb{A}_2$ singular points are nodal	$\frac{12}{7}$
$\mathbb{A}_2$, $\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_2+2\mathbb{A}_1$, $\mathbb{A}_2+3\mathbb{A}_1$, $\mathbb{A}_2+4\mathbb{A}_1$, $2\mathbb{A}_2$, $2\mathbb{A}_2+\mathbb{A}_1$, $2\mathbb{A}_2+2\mathbb{A}_1$, $3\mathbb{A}_2$, $3\mathbb{A}_2+\mathbb{A}_1$, $4\mathbb{A}_2$ some elements of $|-K_X|$ containing $\mathbb{A}_2$ singular points are cuspidal	$\frac{3}{2}$
$\mathbb{A}_3$, $\mathbb{A}_3+\mathbb{A}_1$, $\mathbb{A}_3+2\mathbb{A}_1$, $\mathbb{A}_3+3\mathbb{A}_1$, $\mathbb{A}_3+4\mathbb{A}_1$, $\mathbb{A}_3+\mathbb{A}_2$, $\mathbb{A}_3+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_3+\mathbb{A}_2+2\mathbb{A}_1$, $2\mathbb{A}_3$, $2\mathbb{A}_3+\mathbb{A}_1$, $2\mathbb{A}_3+2\mathbb{A}_1$	$\frac{3}{2}$
$\mathbb{A}_4$, $\mathbb{A}_4+\mathbb{A}_1$, $\mathbb{A}_4+2\mathbb{A}_1$, $\mathbb{A}_4+\mathbb{A}_2$, $\mathbb{A}_4+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_4+\mathbb{A}_3$, $2\mathbb{A}_4$	$\frac{4}{3}$
$\mathbb{A}_5$, $\mathbb{A}_5+\mathbb{A}_1$, $\mathbb{A}_5+2\mathbb{A}_1$, $\mathbb{A}_5+\mathbb{A}_2$, $\mathbb{A}_5+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_5+\mathbb{A}_3$	$\frac{6}{5}$
$\mathbb{A}_6$, $\mathbb{A}_6+\mathbb{A}_1$	$\frac{9}{8}$
$\mathbb{A}_7$, $\mathbb{A}_7+\mathbb{A}_1$ and R irreducible	$\frac{18}{17}$
$\mathbb{A}_7$, $\mathbb{A}_7+\mathbb{A}_1$ and R reducible	1
$\mathbb{A}_8$, $\mathbb{D}_4$, $\mathbb{D}_4+\mathbb{A}_1$, $\mathbb{D}_4+2\mathbb{A}_1$, $\mathbb{D}_4+3\mathbb{A}_1$, $\mathbb{D}_4+4\mathbb{A}_1$,$\mathbb{D}_4+\mathbb{A}_2$, $\mathbb{D}_4+\mathbb{A}_3$, $2\mathbb{D}_4$	1
$\mathbb{D}_5$, $\mathbb{D}_5+\mathbb{A}_1$, $\mathbb{D}_5+2\mathbb{A}_1$, $\mathbb{D}_5+\mathbb{A}_2$, $\mathbb{D}_5+\mathbb{A}_3$	$\frac{6}{7}$
$\mathbb{D}_6$, $\mathbb{D}_6+\mathbb{A}_1$, $\mathbb{D}_6+2\mathbb{A}_1$	$\frac{3}{4}$
$\mathbb{D}_7$	$\frac{2}{3}$
$\mathbb{D}_8$, $\mathbb{E}_6$, $\mathbb{E}_6+\mathbb{A}_1$, $\mathbb{E}_6+\mathbb{A}_2$	$\frac{3}{5}$
$\mathbb{E}_7$, $\mathbb{E}_7+\mathbb{A}_1$	$\frac{3}{7}$
$\mathbb{E}_8$	$\frac{3}{11}$

Table 2.3: δ-invariants of Du Val del Pezzo surfaces of degree 1

This result can be viewed as a generalization of α-invariant computations, which were done by I. Cheltsov, D. Kosta, J, Park and J. Won in a series of papers Cheltsov (2009); Cheltsov and Kosta (2014); Park and Won (2010a, 2010b) since δ and α invariants are related as $3\alpha(X) \geq \delta(X) \geq \frac{3\alpha(X)}{2}$ in case of del Pezzo surfaces. The Du Val del Pezzo surfaces of degree three were listed in Coray and Tsfasman (1988), the singularity types of Du Val del Pezzo surfaces of degrees one and two were listed in Tian and Yau (1983). The results of this article confirm the results in Odaka–Spotti–Sun Odaka et al. (2016) paper and lead to finding new K-stable examples of singular Fano threefolds.

2.2 Proof of Main Theorem via Kento Fujita's formulas

In this paper in order to find δ-invariants of Du Val del Pezzo surfaces we apply Abban–Zhuang theory and use Kento Fujita's formulas. Let X be a Du Val del Pezzo surface, and let S be a minimal resolution of X. For a birational morphism $f\colon \widetilde{X} \to X$ and E be a prime divisor in $\widetilde{X}$ we say that E is a prime divisor over X. If E is f-exceptional, we say that E is an exceptional prime divisor over X. We will denote the subvariety $f(E)$ by $C_X(E)$. Let

$$S_X(E) = \frac{1}{(-K_X)^2}\int_0^{\tau} \mathrm{vol}(f^*(-K_X) - vE)dv \text{ and } A_X(E) = 1 + \mathrm{ord}_E(K_{\widetilde{X}} - f^*(K_X)),$$

where $\tau = \tau(E)$ is the pseudo-effective threshold of E with respect to $-K_X$. Let Q be a point in X. We can define a local δ-invariant and a global δ-invariant of X as

$$\delta_Q(X) = \inf_{\substack{E/X \\ Q \in C_X(E)}} \frac{A_X(E)}{S_X(E)} \text{ and } \delta(X) = \inf_{Q\in X} \delta_Q(X)$$

where the infimum runs over all prime divisors E over the surface X such that $Q \in C_X(E)$. Similarly, for the surface S and a point P on S we define local δ-invariant and a global δ-invariant of S as

$$\delta_P(S) = \inf_{\substack{F/S \\ P \in C_S(F)}} \frac{A_S(F)}{S_S(F)} \text{ and } \delta(S) = \inf_{P\in S} \delta_P(S)$$

where $S_S(F)$ and $A_S(F)$ are defined as $S_X(E)$ and $A_X(E)$ above. It is clear that

$$\delta(X) = \delta(S) \text{ and } \delta_Q(X) = \inf_{P: Q = f(P)} \delta_P(S)$$

We now fix a point P on S and choose a smooth curve A on S containing P. Set

$$\tau(A) = \sup\Big\{ v \in \mathbb{R}_{\geqslant 0} \;\Big|\; \text{the divisor } -K_S - vA \text{ is pseudo-effective}\Big\}.$$

For $v \in [0,\tau]$, let $P(v)$ be the positive part of the Zariski decomposition of the divisor $-K_S - vA$, and let $N(v)$ be its negative part. Then we set

$$S\big(W^A_{\bullet,\bullet};P\big) = \frac{2}{K_S^2}\int_0^{\tau(A)} h(v)dv, \text{ where } h(v) = \big(P(v)\cdot A\big) \times \big(N(v)\cdot A\big)_P + \frac{\big(P(v)\cdot A\big)^2}{2}.$$

It follows from (Araujo et al., 2023, Theorem 1.7.1) that:

$$\delta_P(S) \geqslant \min\left\{\frac{1}{S_S(A)}, \frac{1}{S(W^A_{\bullet,\bullet}, P)}\right\}. \tag{2.2.1}$$

Unfortunately, this approach does not always give us a good estimation. If this is the case, we apply the generalization of this method. Let $\sigma : \widehat{S} \to S$ be a weighted blowup of the point P on S. Suppose, in addition, that $\widehat{S}$ is a Mori Dream space Then

- the σ-exceptional curve $E_P \cong \mathbb{P}^1$ such that $\sigma(E_P) = P$,
- the log pair $(\widehat{S}, E_P)$ has purely log terminal singularities.

Thus, the birational map σ a plt blowup of a point P. Write

$$K_{E_P} + \Delta_{E_P} = \left(K_{\widehat{S}} + E_P\right)\big|_{E_P},$$

where Δ_{E_P} is an effective $\mathbb{Q}$-divisor on E_P known as the different of the log pair $(\widehat{S}, E_P)$. Note that the log pair (E_P, Δ_{E_P}) has at most Kawamata log terminal singularities, and the divisor $-(K_{E_P} + \Delta_{E_P})$ is $\sigma|_{E_P}$-ample.
Let O be a point on E_P. Set

$$\tau(E_P) = \sup\left\{v \in \mathbb{R}_{\geqslant 0} \;\middle|\; \text{the divisor } \sigma^*(-K_S) - vE_P \text{ is pseudo-effective}\right\}.$$

For $v \in [0, \tau]$, let $\widehat{P}(v)$ be the positive part of the Zariski decomposition of the divisor $\sigma^*(-K_S) - vE_P$, and let $\widehat{N}(v)$ be its negative part. Then we set

$$S\left(W^{E_P}_{\bullet,\bullet}; O\right) = \frac{2}{K^2_{\widehat{S}}} \int_0^{\tau(E_P)} \widehat{h}(v)dv, \text{ where } \widehat{h}(v) = \left(\widehat{P}(v) \cdot E_P\right) \times \left(\widehat{N}(v) \cdot E_P\right)_O + \frac{\left(\widehat{P}(v) \cdot E_P\right)^2}{2}.$$

Let $A_{E_P,\Delta_{E_P}}(O) = 1 - \mathrm{ord}_{\Delta_{E_P}}(O)$. It follows from (Araujo et al., 2023, Theorem 1.7.9) and (Araujo et al., 2023, Corollary 1.7.12) that

$$\delta_P(S) \geqslant \min\left\{\frac{A_S(E_P)}{S_S(E_P)}, \inf_{O \in E_P} \frac{A_{E_P,\Delta_{E_P}}(O)}{S\left(W^{E_P}_{\bullet,\bullet}; O\right)}\right\}, \tag{2.2.2}$$

where the infimum is taken over all points $O \in E_P$. Now for all the points P on S we now either values of local δ-invariants or estimations of them. Taking the minimum we compute $\delta(S)$ - the global δ-invariant of S and thus, $\delta(X) = \delta(S)$ - the global δ-invariant of X. We apply this method to minimal resolutions of Du Val del Pezzo surfaces to prove Main Theorem.

Chapter 3

Du Val del Pezzo surfaces of degrees 7 and 8

It was mentioned in (Araujo et al., 2023, Table 2.1) δ-invariants of smooth del Pezzo surfaces of degrees 7 and 8 are given in the following table:

X	$\mathbb{P}^1 \times \mathbb{P}^1$	blowup of $\mathbb{P}^2$ in one point	blowup of $\mathbb{P}^2$ in two points
$\delta(X)$	1	$\frac{6}{7}$	$\frac{21}{25}$

Table 3.1: δ-invariants of Du Val del Pezzo surfaces of degrees 7 and 8

In this chapter, we compute δ-invariants of singular Du Val del Pezzo surfaces of degrees 7 and 8. More precisely we prove that if X is a del Pezzo surface with $\mathbb{A}_1$-singularity of degree 8 then $\delta(X) = \frac{4}{3}$ and if X is a del Pezzo surface with $\mathbb{A}_1$-singularity of degree 7 then $\delta(X) = \frac{21}{31}$.

3.1 Del Pezzo surface of degree 8 with $\mathbb{A}_1$ singularity

Let X be a singular del Pezzo surface of degree 8 with one $\mathbb{A}_1$ singularity. Then X contains 0 lines. Let E be the unique (-2)-curve. One has $\delta_P(X) = \frac{3}{4}$ for $P \in X$. Thus $\delta(X) = \frac{3}{4}$.

Proof. Let F be a fiber of natural projection on $\mathbb{P}^1$. Then $\tau(F) = 4$ and the Zariski Decomposition of the divisor $-K_X - vF$ is given by:

$$P(v) = -K_X - vF - \frac{v}{2}E \text{ and } N(v) = \frac{v}{2}E \text{ if } v \in [0,4].$$

Moreover,

$$(P(v))^2 = \frac{(4-v)^2}{2} \text{ and } P(v) \cdot F = 2 - \frac{v}{2} \text{ if } v \in [0,4].$$

We have $S_X(F) = \frac{4}{3}$. Thus, $\delta_P(S) \leq \frac{3}{4}$. Note that $h(v) \leq \frac{(4-v)(4+v)}{8}$ if $v \in [0,4]$. So $S(W^F_{\bullet,\bullet};P) \leq \frac{4}{3}$. Thus, $\delta_P(X) = \frac{3}{4}$ for $P \in X$ and $\delta(X) = \frac{3}{4}$. □

3.2 Del Pezzo surface of degree 7 with $\mathbb{A}_1$ singularity

Let X be a singular del Pezzo surface with one $\mathbb{A}_1$ singularity, S be a minimal resolution of X. Then X contains 2 lines, and S can be obtained by blowing up $\mathbb{P}^2$ at point P_1; and a point P_2 on the exceptional divisor corresponding to P_1. Let E_1, E_2 be the exceptional divisors corresponding to P_1, P_2; L_{12} be a (-1)-curve which is a strict transform of the line passing through P_1. The dual graph of (-1) and (-2) curves is given in the following picture:

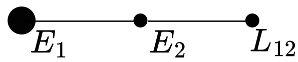


Figure 3.1: Dual graph: $(-K_S)^2 = 7$, singularity $\mathbb{A}_1$

One has

$$\delta_P(S) = \begin{cases} \frac{21}{31} \text{ if } P \in E_2, \\ \frac{7}{9} \text{ if } P \in E_1 \backslash E_2, \\ \frac{21}{25} \text{ if } P \in L_{12} \backslash E_2. \end{cases} \quad \text{and } \delta_P(S) \geq \frac{21}{23} \text{ otherwise.}$$

Thus $\delta_P(X) = \frac{21}{31}$.

Proof. **Step 1.** Suppose $P \in L_{12}$. Then $\tau(L_{12}) = 3$ and the Zariski Decomposition of the divisor $-K_S - vL_{12}$ is given by:

$$P(v) = \begin{cases} -K_S - vL_{12} \text{ if } v \in [0,1], \\ -K_S - vL_{12} - (v-1)(E_1 + 2E_2) \text{ if } v \in [1,3]. \end{cases}$$

$$N(v) = \begin{cases} 0 \text{ if } v \in [0,1], \\ (v-1)(E_1 + 2E_2) \text{ if } v \in [1,3]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 7 - 2v - v^2 \text{ if } v \in [0,1], \\ (3-v)^2 \text{ if } v \in [1,3]. \end{cases} \quad P(v) \cdot L_{12} = \begin{cases} 1 + v \text{ if } v \in [0,1], \\ 3 - v \text{ if } v \in [1,3]. \end{cases}$$

We have $S_S(L_{12}) = \frac{25}{21}$. Thus, $\delta_P(S) \leq \frac{21}{25}$ for $P \in L_{12}$. Note that for $P \in L_{12} \backslash E_2$ we have:

$$h(v) \leq \begin{cases} \frac{(1+v)^2}{2} \text{ if } v \in [0,1], \\ \frac{(3-v)^2}{2} \text{ if } v \in [1,3]. \end{cases}$$

So $S(W^{L_{12}}_{\bullet,\bullet}; P) \leq \frac{5}{7} < \frac{25}{21}$. Thus, $\delta_P(S) = \frac{21}{25}$ if $P \in L_{12} \backslash E_2$.

Step 2. Suppose $P \in E_1$. Then $\tau(E_1) = 2$ and the Zariski Decomposition of the divisor $-K_S - vE_1$ is given by:

$$P(v) = \begin{cases} -K_S - vE_1 \text{ if } v \in [0,1], \\ -K_S - vE_1 - (v-1)E_2 \text{ if } v \in [1,2]. \end{cases} \qquad N(v) = \begin{cases} 0 \text{ if } v \in [0,1], \\ (v-1)E_2 \text{ if } v \in [1,2]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 7 - 2v^2 \text{ if } v \in [0,1], \\ (2-v)(4+v) \text{ if } v \in [1,2]. \end{cases} \qquad P(v) \cdot E_1 = \begin{cases} 2v \text{ if } v \in [0,1], \\ v+1 \text{ if } v \in [1,2]. \end{cases}$$

We have $S_S(E_1) = \frac{9}{7}$. Thus, $\delta_P(S) \leq \frac{7}{9}$ for $P \in E_1$. Note that for $P \in E_1 \backslash E_2$

$$h(v) \leq \begin{cases} 2v^2 \text{ if } v \in [0,1], \\ \frac{(v+1)^2}{2} \text{ if } v \in [1,2]. \end{cases}$$

So $S(W^{E_1}_{\bullet,\bullet}; P) \leq \frac{23}{21} < \frac{9}{7}$. Thus, $\delta_P(S) = \frac{7}{9}$ if $P \in E_1 \backslash E_2$.

Step 3. Suppose $P \in E_2$. Then $\tau(E_2) = 4$ and Zariski Decomposition of the divisor $-K_S - vE_2$ is given by:

$$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{2}E_1 \text{ if } v \in [0,1], \\ -K_S - vE_2 - \frac{v}{2}E_1 - (v-1)L_{12} \text{ if } v \in [1,4]. \end{cases} \qquad N(v) = \begin{cases} \frac{v}{2}E_1 \text{ if } v \in [0,1], \\ \frac{v}{2}E_1 + (v-1)L_{12} \text{ if } v \in [1,4]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 7 - 2v - \frac{v^2}{2} \text{ if } v \in [0,1], \\ \frac{(4-v)^2}{2} \text{ if } v \in [1,4]. \end{cases} \qquad P(v) \cdot E_2 = \begin{cases} 1 + \frac{v}{2} \text{ if } v \in [0,1], \\ 2 - \frac{v}{2} \text{ if } v \in [1,4]. \end{cases}$$

We have $S_S(E_2) = \frac{31}{21}$. Thus, $\delta_P(S) \leq \frac{21}{31}$ for $P \in E_2$. Note that if $P \in E_2 \backslash L_{12}$ or if $P \in E_2 \cap L_{12}$ then

$$h(v) \leq \begin{cases} \frac{(v+2)(3v+2)}{8} \text{ if } v \in [0,1], \\ \frac{(4-v)(v+4)}{8} \text{ if } v \in [1,4]. \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{(2+v)^2}{8} \text{ if } v \in [0,1], \\ \frac{3(4-v)v}{8} \text{ if } v \in [1,4]. \end{cases}$$

So $S(W^{E_2}_{\bullet,\bullet}; P) \leq \frac{17}{21} < \frac{31}{21}$ or $S(W^{E_2}_{\bullet,\bullet}; P) \leq \frac{13}{21} \leq \frac{31}{21}$. Thus, $\delta_P(S) = \frac{21}{31}$ if $P \in E_2$.

Step 4. Suppose $P \notin E_1 \cup E_2 \cup L_2$. Consider a blowup $\pi : \widetilde{S} \to S$ at point P with the exceptional divisor E_P. Suppose $\widetilde{E}_1$ is a strict transform of E_1 and L_{1P} is a strict transform of the line through P_1 and a projection of P on $\mathbb{P}^2$. Then $\tau(E_P) = 3$ and Zariski Decomposition of the divisor $\sigma^*(-K_S) - vE_P$ is given by:

$$P(v) = \begin{cases} \sigma^*(-K_S) - vE_P \text{ if } v \in [0,2], \\ \sigma^*(-K_S) - vE_P - (v-2)(2L_{1P} + \widetilde{E}_1) \text{ if } v \in [2,3]. \end{cases}$$

$$N(v) = \begin{cases} 0 \text{ if } v \in [0,2], \\ (v-2)(2L_{1P} + \widetilde{E}_1) \text{ if } v \in [2,3]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 7 - v^2 \text{ if } v \in [0,2], \\ (3-v)(5-v) \text{ if } v \in [2,3]. \end{cases} \qquad P(v) \cdot E_P = \begin{cases} v \text{ if } v \in [0,2], \\ 4 - v \text{ if } v \in [2,3]. \end{cases}$$

We have $S_S(E_P) = \frac{38}{21}$. Thus, $\delta_P(S) \leq \frac{2}{38/21} = \frac{21}{19}$. Note that for $O \in E_P$ we have:

$$h(v) = \begin{cases} \frac{v^2}{2} \text{ if } v \in [0,2], \\ \frac{(4-v)(3v-4)}{2} \text{ if } v \in [2,3]. \end{cases}$$

So $S(W^{E_P}_{\bullet,\bullet}; O) \leq \frac{23}{21}$. Thus, $\delta_P(S) \leq \frac{21}{23}$ if $P \notin E_1 \cup E_2 \cup L_{12}$. $\square$

Chapter 4

Du Val Del Pezzo surfaces of degree 6

It was mentioned in (Araujo et al., 2023, Table 2.1) that $\delta(X) = 1$ when X is a smooth del Pezzo surface of degree 6. In this chapter, we compute δ-invariants of singular Du Val del Pezzo surfaces of degree 6.

MAIN THEOREM Let X be a singular del Pezzo surface of degree 6. Then the δ-invariant of X is uniquely determined by the degree of X, the number of lines on X, and the type of singularities on X which is given in the following table:

K_X^2	# lines	Sing(X)	$\delta(X)$
6	3	$\mathbb{A}_1$	$\frac{3}{4}$
6	4	$\mathbb{A}_1$	$\frac{9}{11}$
6	2	$2\mathbb{A}_1$	$\frac{9}{14}$
6	2	$\mathbb{A}_2$	$\frac{3}{5}$
6	1	$\mathbb{A}_2+\mathbb{A}_1$	$\frac{1}{2}$

Table 4.1: δ-invariants of Du Val del Pezzo surfaces of degree 6

4.1 General results for degree 6

Let X be a del Pezzo surface of degree 6 with at most Du Val singularities, S be a minimal resolution of X and P be a point on S. Then:

Lemma 4.1.1. *Let P be a general point on S. Consider the blowup $\sigma : \widetilde{S} \to S$ of S at P with the exceptional divisor A there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:*

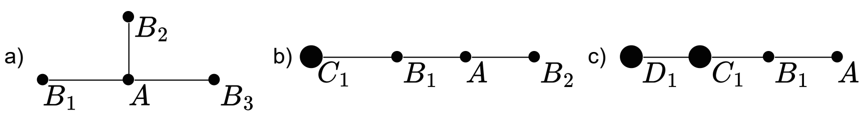


Figure 4.1: Dual graph: $(-K_S)^2 = 6$ for a general point

Then $\tau(A) = 3$ and the Zariski Decomposition of the divisor $\sigma^*(-K_S) - vA$ is given by:

a).

$$P(v) = \begin{cases} \sigma^*(-K_S) - vA \text{ if } v \in [0,2], \\ \sigma^*(-K_S) - vA - (v-2)(B_1 + B_2 + B_3) \text{ if } v \in [2,3]. \end{cases}$$

$$N(v) = \begin{cases} 0 \text{ if } v \in [0,2], \\ (v-2)(B_1 + B_2 + B_3) \text{ if } v \in [2,3]. \end{cases}$$

b).

$$P(v) = \begin{cases} \sigma^*(-K_S) - vA \text{ if } v \in [0,2], \\ \sigma^*(-K_S) - vA - (v-2)(2B_1 + C_1 + B_2) \text{ if } v \in [2,3]. \end{cases}$$

$$N(v) = \begin{cases} 0 \text{ if } v \in [0,2], \\ (v-2)(2B_1 + C_1 + B_2) \text{ if } v \in [2,3]. \end{cases}$$

c).

$$P(v) = \begin{cases} \sigma^*(-K_S) - vA \text{ if } v \in [0,2], \\ \sigma^*(-K_S) - vA - (v-2)(3B_1 + 2C_1 + D_1) \text{ if } v \in [2,3]. \end{cases}$$

$$N(v) = \begin{cases} 0 \text{ if } v \in [0,2], \\ (v-2)(3B_1 + 2C_1 + D_1) \text{ if } v \in [2,3]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 6 - v^2 \text{ if } v \in [0,2], \\ 2(3-v)^2 \text{ if } v \in [2,3]. \end{cases} \quad P(v)\cdot A = \begin{cases} v \text{ if } v \in [0,2], \\ 2(3-v) \text{ if } v \in [2,3]. \end{cases}$$

In this case $\delta_P(S) \geq 1$.

Proof. The Zariski Decomposition in part a). follows from $\sigma^*(-K_S) - vA \sim (3-v)A + B_1 + B_2 + B_3$. A similar statement holds in other parts. We have $S_S(A) = \frac{5}{3}$. Thus, $\delta_P(S) \leq \frac{2}{5/3} = \frac{6}{5}$. Moreover,

$$h(v) = \begin{cases} \frac{v^2}{2} \text{ if } v \in [0,2], \\ 2(3-v)(2v-3) \text{ if } v \in [2,3]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; O) \leq 1$. We get that $\delta_P(S) \geq 1$. □

Now we consider a curve A on S. Small circles correspond to a (-1)-curves and large circles correspond to a (-2)-curves on dual graphs.

Lemma 4.1.2. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph: Then $\tau(A) = 2$ and the Zariski Decomposition of*

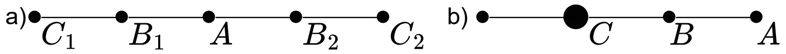


Figure 4.2: Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = 1$

the divisor $-K_S - vA$ is given by:

a).
$$P(v) = \begin{cases} -K_S - vA \text{ if } v \in [0,1], \\ -K_S - vA - (v-1)(B_1 + B_2) \text{ if } v \in [1,2]. \end{cases}$$
$$N(v) = \begin{cases} 0 \text{ if } v \in [0,1], \\ (v-1)(B_1 + B_2) \text{ if } v \in [1,2]. \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vA \text{ if } v \in [0,1], \\ -K_S - vA - (v-1)(2B + C) \text{ if } v \in [1,2]. \end{cases}$$
$$N(v) = \begin{cases} 0 \text{ if } v \in [0,1], \\ (v-1)(2B + C) \text{ if } v \in [1,2]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 6 - 2v - v^2 \text{ if } v \in [0,1], \\ (2-v)(4-v) \text{ if } v \in [1,2]. \end{cases} \quad P(v) \cdot A = \begin{cases} v + 1 \text{ if } v \in [0,1], \\ 3 - v \text{ if } v \in [1,2]. \end{cases}$$

In this case $\delta_P(S) = 1$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} \big(2 - v\big)A + \frac{1}{2}\big(3B_1 + C_1 + 3B_2 + C_2\big)$. A similar statement holds in other parts. We have $S_S(A) = 1$. Thus, $\delta_P(S) \leq 1$ for $P \in A$. Moreover, for $P \in A \backslash B$:

$$h(v) \leq \begin{cases} \frac{(v+1)^2}{2} \text{ if } v \in [0,1], \\ \frac{(3-v)(v+1)}{2} \text{ if } v \in [1,2]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \leq 1$. We get that $\delta_P(S) = 1$ for $P \in A \backslash B$. □

Lemma 4.1.3. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

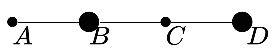


Figure 4.3: Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = \frac{9}{10}$

Then $\tau(A)=2$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

$$P(v)=-K_S-vA-\frac{v}{2}B \text{ and } N(v)=\frac{v}{2}B \text{ if } v\in[0,2].$$

Moreover,

$$(P(v))^2=\frac{(2-v)(6+v)}{2} \text{ and } P(v)\cdot A=1+\frac{v}{2} \text{ if } v\in[0,2].$$

In this case $\delta_P(S)=\frac{9}{10}$ if $P\in A\backslash B$.

Proof. The Zariski Decomposition follows from $-K_S-vA\sim_{\mathbb{R}}\big(2-v\big)A+3B+4C+2D$. We have $S_S(A)=\frac{10}{9}$. Thus, $\delta_P(S)\leq\frac{9}{10}$ for $P\in A$. Note that if $P\in A\backslash B$ we have $h(v)=\frac{(2+v)^2}{8}$ if $v\in[0,2]$. So $S(W^A_{\bullet,\bullet};P)\leq\frac{7}{9}\leq\frac{10}{9}$. Thus, $\delta_P(S)=\frac{9}{10}$ if $P\in A\backslash B$. □

Lemma 4.1.4. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph and no other (-1)-curves and (-2)-curves intersect A:*

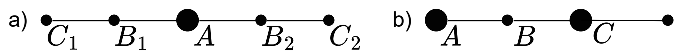


Figure 4.4: Dual graph: $(-K_S)^2=6$ and $\delta_P(S)=\frac{9}{11}$ with $\tau(A)=2$

Then $\tau(A)=2$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

a).

$$P(v)=\begin{cases}-K_S-vA \text{ if } v\in[0,1],\\ -K_S-vA-(v-1)(B_1+B_2) \text{ if } v\in[1,2].\end{cases}$$

$$N(v)=\begin{cases}0 \text{ if } v\in[0,1],\\ (v-1)(B_1+B_2) \text{ if } v\in[1,2].\end{cases}$$

b).

$$P(v)=\begin{cases}-K_S-vA \text{ if } v\in[0,1],\\ -K_S-vA-(v-1)(2B_1+C) \text{ if } v\in[1,2].\end{cases}$$

$$N(v)=\begin{cases}0 \text{ if } v\in[0,1],\\ (v-1)(2B_1+C) \text{ if } v\in[1,2].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}6-2v^2 \text{ if } v\in[0,1],\\ 8-4v \text{ if } v\in[1,2].\end{cases}\quad P(v)\cdot A=\begin{cases}2v \text{ if } v\in[0,1],\\ 2 \text{ if } v\in[1,2].\end{cases}$$

In this case $\delta_P(S)=\frac{9}{11}$ if $P\in A\backslash B$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} \big(2-v\big)A + 2B_1 + C_1 + 2B_2 + C_2$. A similar statement holds in other parts. We have $S_S(A) = \frac{11}{9}$. Thus, $\delta_P(S) \leq \frac{9}{11}$ for $P \in A \backslash B$. Moreover,

$$h(v) = \begin{cases} 2v^2 \text{ if } v \in [0,1], \\ 2v \text{ if } v \in [1,2]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{11}{9}$. We get that $\delta_P(S) = \frac{9}{11}$ for $P \in A \backslash B$. □

Lemma 4.1.5. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

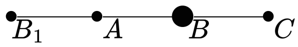


Figure 4.5: Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = \frac{9}{11}$ with $\tau(A) = 3$

Then $\tau(A) = 3$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B - (v-1)B_1 \text{ if } v \in [1,2], \\ -K_S - vA - (v-1)(B+B_1) - (v-2)C \text{ if } v \in [2,3], \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}B \text{ if } v \in [0,1], \\ \frac{v}{2}B + (v-1)B_1 \text{ if } v \in [1,2], \\ (v-1)(B+B_1) + (v-2)C \text{ if } v \in [2,3]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} \frac{(2-v)(6+v)}{2} \text{ if } v \in [0,1], \\ \frac{v^2}{2} - 4v + 7 \text{ if } v \in [1,2], \\ (3-v)^2 \text{ if } v \in [2,3]. \end{cases} \quad P(v) \cdot A = \begin{cases} 1 + \frac{v}{2} \text{ if } v \in [0,1], \\ 2 - \frac{v}{2} \text{ if } v \in [1,2], \\ 3 - v \text{ if } v \in [2,3]. \end{cases}$$

In this case $\delta_P(S) = \frac{9}{11}$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} \big(3-v\big)A + 2B_1 + 2B + C$. We have $S_S(A) = \frac{11}{9}$. Thus, $\delta_P(S) \leq \frac{9}{11}$ for $P \in A$. Moreover, if $P \in A \backslash B$:

$$h(v) \leq \begin{cases} \frac{(v+2)^2}{8} \text{ if } v \in [0,1], \\ \frac{3(4-v)v}{8} \text{ if } v \in [1,2], \\ \frac{(3-v)(v+1)}{2} \text{ if } v \in [2,3]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq 1 \leq \frac{11}{9}$. We get that $\delta_P(S) = \frac{9}{11}$ for $P \in A \backslash B$. □

Lemma 4.1.6. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

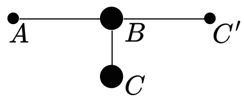


Figure 4.6: Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = \frac{4}{5}$

Then $\tau(A) = 3$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2B+C) \text{ if } v \in \big[0, \frac{3}{2}\big], \\ -K_S - vA - (v-1)(2B+C) - (2v-3)C' \text{ if } v \in \big[\frac{3}{2}, 3\big]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2B+C) \text{ if } v \in \big[0, \frac{3}{2}\big], \\ (v-1)(2B+C) + (2v-3)C' \text{ if } v \in \big[\frac{3}{2}, 3\big]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 6 - 2v - \frac{v^2}{3} \text{ if } v \in \big[0, \frac{3}{2}\big], \\ (3-v)^2 \text{ if } v \in \big[\frac{3}{2}, 3\big]. \end{cases} \quad P(v) \cdot A = \begin{cases} 1 + \frac{v}{3} \text{ if } v \in \big[0, \frac{3}{2}\big], \\ 3 - v \text{ if } v \in \big[\frac{3}{2}, 3\big]. \end{cases}$$

In this case $\delta_P(S) = \frac{4}{5}$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} \big(3-v\big)A + 4B + 2C + 3C'$. We have $S_S(A) = \frac{5}{4}$. Thus, $\delta_P(S) \le \frac{4}{5}$ for $P \in A$. Moreover, if $P \in A \backslash B$:

$$h(v) = \begin{cases} \frac{(3+v)^2}{18} \text{ if } v \in \big[0, \frac{3}{2}\big], \\ \frac{(3-v)^2}{2} \text{ if } v \in \big[\frac{3}{2}, 3\big]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \le \frac{7}{12} \le \frac{5}{4}$. We get that $\delta_P(S) = \frac{4}{5}$ for $P \in A \backslash B$. □

Lemma 4.1.7. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

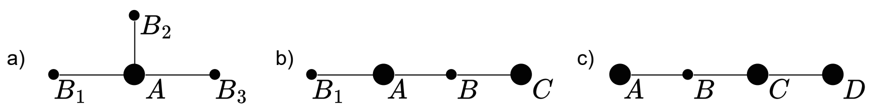


Figure 4.7: Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = \frac{3}{4}$ with $\tau(A) = 3$

Then $\tau(A) = 3$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a). $$P(v) = \begin{cases} -K_S - vA \text{ if } v \in [0,1], \\ -K_S - vA - (v-1)(B_1 + B_2 + B_3) \text{ if } v \in [1,3]. \end{cases}$$

$$N(v) = \begin{cases} 0 \text{ if } v \in [0,1], \\ (v-1)(B_1+B_2+B_3) \text{ if } v \in [1,3]. \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vA \text{ if } v \in [0,1], \\ -K_S - vA - (v-1)(B_1+2B+C) \text{ if } v \in [1,3]. \end{cases}$$

$$N(v) = \begin{cases} 0 \text{ if } v \in [0,1], \\ (v-1)(B_1+2B+C) \text{ if } v \in [1,3]. \end{cases}$$

c).
$$P(v) = \begin{cases} -K_S - vA \text{ if } v \in [0,1], \\ -K_S - vA - (v-1)(3B+2C+D) \text{ if } v \in [1,3]. \end{cases}$$

$$N(v) = \begin{cases} 0 \text{ if } v \in [0,1], \\ (v-1)(3B+2C+D) \text{ if } v \in [1,3]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2(3-v^2) \text{ if } v \in [0,1], \\ (3-v)^2 \text{ if } v \in [1,3]. \end{cases} \quad P(v) \cdot A = \begin{cases} 2v \text{ if } v \in [0,1], \\ 3-v \text{ if } v \in [1,3]. \end{cases}$$

In this case $\delta_P(S) = \frac{3}{4}$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} \big(3-v\big)A + 2B_1 + 2B_2 + 2B_3$. A similar statement holds in other parts. We have $S_S(A) = \frac{4}{3}$. Thus, $\delta_P(S) \leq \frac{3}{4}$ for $P \in A$. Note that if $P \in A \backslash B$ we have:

$$h(v) \leq \begin{cases} 2v^2 \text{ if } v \in [0,1], \\ \frac{(3-v)(v+1)}{2} \text{ if } v \in [1,3]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{10}{9} \leq \frac{4}{3}$. Thus, $\delta_P(S) = \frac{3}{4}$ if $P \in A \backslash B$. ☐

Lemma 4.1.8. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

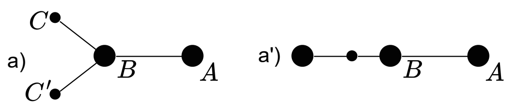


Figure 4.8: Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = \frac{3}{4}$ with $\tau(A) = 2$

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = -K_S - vA - \frac{v}{2}B \text{ and } N(v) = \frac{v}{2}B \text{ if } v \in [0,2].$$

Moreover,

$$(P(v))^2 = \frac{3(2-v)(6+v)}{2} \text{ and } P(v)\cdot A = \frac{3v}{2} \text{ if } v \in [0,2].$$

In this case $\delta_P(S) = \frac{3}{4}$ *if* $P \in A\backslash B$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} \big(2-v\big)A + 4B + 3C + 3C'$. We have $S_S(A) = \frac{4}{3}$. Thus, $\delta_P(S) \leq \frac{3}{4}$ for $P \in A$. Note that if $P \in A\backslash B$ we have $h(v) = \frac{9v^2}{8}$ if $v \in [0,2]$. So $S(W^A_{\bullet,\bullet};P) \leq \frac{4}{3}$. Thus, $\delta_P(S) = \frac{3}{4}$ if $P \in A\backslash B$. □

Lemma 4.1.9. *Suppose* P *belongs to a* (-1)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

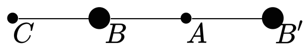


Figure 4.9: Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = \frac{9}{14}$

Then $\tau(A) = 4$ *and the Zariski Decomposition of the divisor* $-K_S - vA$ *is given by:*

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}(B+B') \text{ if } v \in [0,2], \\ -K_S - vA - \frac{v}{2}B' - (v-1)B - (v-2)C \text{ if } v \in [2,4]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(B+B') \text{ if } v \in [0,2], \\ \frac{v}{2}B' + (v-1)B + (v-2)C \text{ if } v \in [2,4]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2(3-v) \text{ if } v \in [0,2], \\ \frac{(v-4)^2}{2} \text{ if } v \in [2,4]. \end{cases} \quad P(v)\cdot A = \begin{cases} 1 \text{ if } v \in [0,2], \\ 2 - \frac{v}{2} \text{ if } v \in [2,4]. \end{cases}$$

In this case $\delta_P(S) = \frac{9}{14}$ *if* $P \in A$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} \big(4-v\big)A + 3B + 2C + 2B'$. We have $S_S(A) = \frac{14}{9}$. Thus, $\delta_P(S) \leq \frac{9}{14}$ for $P \in A$. Moreover if $P \in A \cap B'$ or $P \in A\backslash B'$:

$$h(v) = \begin{cases} \frac{v+1}{2} \text{ if } v \in [0,2], \\ \frac{(4-v)(4+v)}{8} \text{ if } v \in [2,4]. \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{v+1}{2} \text{ if } v \in [0,2], \\ \frac{3(4-v)v}{8} \text{ if } v \in [2,4]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{11}{9} \leq \frac{14}{9}$ or $S(W^A_{\bullet,\bullet};P) \leq \frac{4}{3} \leq \frac{14}{9}$. We get that $\delta_P(S) = \frac{9}{14}$ for $P \in A$. □

Lemma 4.1.10. *Suppose* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

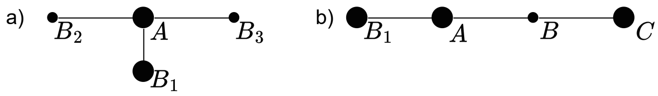


Figure 4.10: Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = \frac{3}{5}$

Then $\tau(A) = 4$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B_1 - (v-1)(B_2+B_3) \text{ if } v \in [1,4]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ \frac{v}{2}B_1 - (v-1)(B_2+B_3) \text{ if } v \in [1,4]. \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B_1 - (v-1)(2B+C) \text{ if } v \in [1,4]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ \frac{v}{2}B_1 - (v-1)(2B+C) \text{ if } v \in [1,4]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} \frac{3(2-v)(2+v)}{2} \text{ if } v \in [0,1], \\ \frac{(4-v)^2}{2} \text{ if } v \in [1,4]. \end{cases} \quad P(v)\cdot A = \begin{cases} \frac{3v}{2} \text{ if } v \in [0,1], \\ 2 - \frac{v}{2} \text{ if } v \in [1,4]. \end{cases}$$

In this case $\delta_P(S) = \frac{3}{5}$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} \big(4-v\big)A + 2B_1 + 3B_2 + 3B_3$. A similar statement holds in other parts. We have $S_S(A) = \frac{5}{3}$. Thus, $\delta_P(S) \le \frac{3}{5}$ for $P \in E_2$. Note that if $P \in A \backslash B_1$ or if $P \in A \cap B_1$: then:

$$h(v) \le \begin{cases} \frac{9v^2}{8} \text{ if } v \in [0,1], \\ \frac{3(4-v)v}{8} \text{ if } v \in [1,4]. \end{cases} \quad \text{or } h(v) \le \begin{cases} \frac{15v^2}{8} \text{ if } v \in [0,1], \\ \frac{(4-v)(4+v)}{8} \text{ if } v \in [1,4]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \le \frac{5}{4} \le \frac{5}{3}$ or $S(W^A_{\bullet,\bullet};P) \le \frac{4}{3} \le \frac{5}{3}$. Thus, $\delta_P(S) = \frac{3}{5}$ if $P \in A \backslash B$. □

Lemma 4.1.11. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

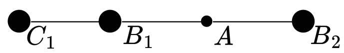


Figure 4.11: Dual graph: $(-K_S)^2 = 6$ and $\delta_P(S) = \frac{1}{2}$

Then $\tau(A) = 6$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = -K_S - vA - \frac{v}{3}(2B_1 + C_1) - \frac{v}{2}B_2 \text{ and } N(v) = \frac{v}{3}(2B_1 + C_1) + \frac{v}{2}B_2 \text{ if } v \in [0,6].$$

Moreover,

$$(P(v))^2 = \frac{(6-v)^2}{2} P(v) \cdot A = 1 - \frac{v}{6} \text{ if } v \in [0,6].$$

In this case $\delta_P(S) = \frac{1}{2}$ if $P \in A$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} \big(6-v\big)A + 4B_1 + 2C_1 + 3B_2$. We have $S_S(A) = 2$. Thus, $\delta_P(S) \le \frac{1}{2}$ for $P \in A$. Note that $h(v) \le \frac{(6-v)(7v+6)}{72}$ if $v \in [0,6]$. So $S(W^A_{\bullet,\bullet}; P) \le \frac{5}{3} \le 2$. Thus, $\delta_P(S) = \frac{1}{2}$ if $P \in A$. □

4.2 Finding δ-invariants for degree 6

Let X be a singular del Pezzo surface of degree 6 with and S be a minimal resolution of X. Then there are several possible cases:

I. X has an $\mathbb{A}_1$ singularity and contains 3 lines. In this case, we let E be the exceptional divisor, L_i for $i \in \{1,2,3\}$ are the lines on S,

II. X has an $\mathbb{A}_1$ singularity and contains 4 lines. In this case, we let E be the exceptional divisor, L_i and L_i' for $i \in \{1,2\}$ be the lines on S,

III. X has two $\mathbb{A}_1$ singularities and contains 2 lines. In this case, we let E_i for $i \in \{1,2\}$ be the exceptional divisors, L_1 and L_{12} are the lines on S,

IV. X has an $\mathbb{A}_2$ singularity and contains 2 lines. In this case, we let E_i for $i \in \{1,2\}$ be the exceptional divisors, L_1 and L_1' be the lines on S,

V. X has $\mathbb{A}_2$ and $\mathbb{A}_1$ singularities and contains 1 line. In this case, we let E_i for $i \in \{1,2,3\}$ be the exceptional divisors, L_{13} be the line on S.

such that the dual graph of the (-1)-curves and (-2)-curves on S is given the picture below.

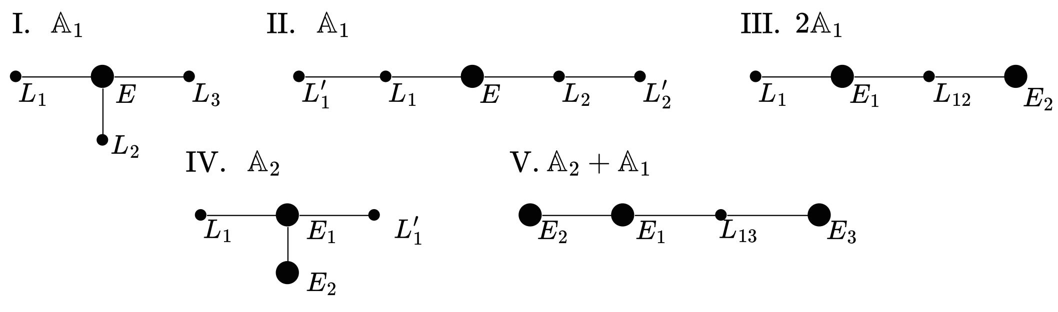


Figure 4.12: Du Val del Pezzo surfaces with $(-K_S)^2 = 6$

Then

I. $\delta(X) = \frac{3}{4}$ since depending on the position of point $P \in S$ we have

P	E	$(L_1 \cup L_2 \cup L_3) \backslash E$	o/w
$\delta_P(S)$	$\frac{3}{4}$	$\frac{9}{10}$	≥ 1

Table 4.2: Local δ-invariants: $(-K_S)^2 = 6$ and $\mathbb{A}_1$ singularity, 3 lines

II. $\delta(X) = \frac{9}{11}$ since since depending on the position of point $P \in S$ we have

P	$E \cup L_1 \cup L_2$	$(L_1' \cup L_2') \backslash (L_1 \cup L_2)$	o/w
$\delta_P(S)$	$\frac{9}{10}$	1	≥ 1

Table 4.3: Local δ-invariants: $(-K_S)^2 = 6$ and $\mathbb{A}_1$ singularity, 4 lines

III. $\delta(X) = \frac{9}{14}$ since since depending on the position of point $P \in S$ we have

P	L_{12}	$E_1 \backslash L_{12}$	$E_2 \backslash L_{12}$	$L_1 \backslash E_1$	o/w
$\delta_P(S)$	$\frac{9}{14}$	$\frac{3}{4}$	$\frac{9}{11}$	$\frac{9}{10}$	≥ 1

Table 4.4: Local δ-invariants: $(-K_S)^2 = 6$ and $2\mathbb{A}_1$ singularities

IV. $\delta(X) = \frac{3}{5}$ since since depending on the position of point $P \in S$ we have

P	E_1	$E_2 \backslash E_1$	$(L_1 \cup L_1') \backslash E_1$	o/w
$\delta_P(S)$	$\frac{3}{5}$	$\frac{3}{4}$	$\frac{4}{5}$	≥ 1

Table 4.5: Local δ-invariants: $(-K_S)^2 = 6$ and $\mathbb{A}_2$ singularity

V. $\delta(X) = \frac{1}{2}$ since since depending on the position of point $P \in S$ we have

P	L_{13}	$E_1 \backslash L_{13}$	$(E_2 \cup E_3) \backslash (E_1 \backslash L_{13})$	o/w
$\delta_P(S)$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{3}{4}$	≥ 1

Table 4.6: Local δ-invariants: $(-K_S)^2 = 6$ and $\mathbb{A}_2\mathbb{A}_1$ singularities

Proof. We prove each case separately using lemmas from the previous section.

I. If $P \in E$ the assertion follows from Lemma 4.1.7. If $P \in (L_1 \cup L_2 \cup L_3) \backslash E$ the assertion follows from Lemma 4.1.3. If P is a general point the assertion follows from Lemma 4.1.1.

II. If $P \in E$ the assertion follows from Lemma 4.1.4 [a).]. If $P \in (L_1 \cup L_2) \backslash E$ the assertion follows from Lemma 4.1.5. If $P \in (L_1' \cup L_2') \backslash (L_1 \cup L_2)$ the assertion follows from Lemma 4.1.2 [b).]. If P is a general point the assertion follows from Lemma 4.1.1.

III. If $P \in L_{12}$ the assertion follows from Lemma 4.1.9. If $P \in E_1 \backslash L_{12}$ the assertion follows from Lemma 4.1.7 [b).]. If $P \in E_2 \backslash L_{12}$ the assertion follows from Lemma 4.1.4 [b).]. If $P \in L_1 \backslash E_1$ the assertion follows from Lemma 4.1.3. If P is a general point the assertion follows from Lemma 4.1.1.

IV. If $P \in E_1$ the assertion follows from Lemma 4.1.10 [a).]. If $P \in E_2 \backslash E_1$ the assertion follows from Lemma 4.1.8. If $P \in (L_1 \cup L_1') \backslash E_1$ the assertion follows from Lemma 4.1.6. If P is a general point the assertion follows from Lemma 4.1.1.

V. If $P \in L_{12}$ the assertion follows from Lemma 4.1.11. If $P \in E_1 \backslash L_{13}$ the assertion follows from Lemma 4.1.10 [b).]. If $P \in E_2 \backslash E_1$ the assertion follows from Lemma 4.1.8. If $P \in E_2 \backslash L_{13}$ the assertion follows from Lemma 4.1.7. If P is a general point the assertion follows from Lemma 4.1.1.

□

Chapter 5

Du Val Del Pezzo surfaces of degree 5

In (Araujo et al., 2023, Lemma 2.11) it was proven that $\delta(X) = \frac{15}{13}$ when X is a smooth del Pezzo surface of degree 5. In this chapter, we compute δ-invariants of singular Du Val del Pezzo surfaces of degree 5.

MAIN THEOREM Let X be a singular Du Val del Pezzo surface of degree 5. Then the δ-invariant of X is uniquely determined by the degree of X, the number of lines on X, and the type of singularities on X which is given in the following table:

K_X^2	# lines	$\mathrm{Sing}(X)$	$\delta(X)$	K_X^2	# lines	$\mathrm{Sing}(X)$	$\delta(X)$
5	7	$\mathbb{A}_1$	$\frac{15}{17}$	5	3	$\mathbb{A}_2+\mathbb{A}_1$	$\frac{15}{23}$
5	5	$2\mathbb{A}_1$	$\frac{15}{19}$	5	2	$\mathbb{A}_3$	$\frac{5}{9}$
5	4	$\mathbb{A}_2$	$\frac{5}{7}$	5	1	$\mathbb{A}_4$	$\frac{3}{7}$

Table 5.1: δ-invariants of Du Val del Pezzo surfaces of degree 5

5.1 General results for degree 5

Let X be a del Pezzo surface of degree 5 with at most Du Val singularities, S be a minimal resolution of X and P is a point on S. Then:

Lemma 5.1.1. *Suppose is a general point on S. Consider the blowup $\sigma : \widetilde{S} \to S$ of S at P with the exceptional divisor A. There exist (-1)-curves and (-2)-curves which form one of the following dual graphs:*

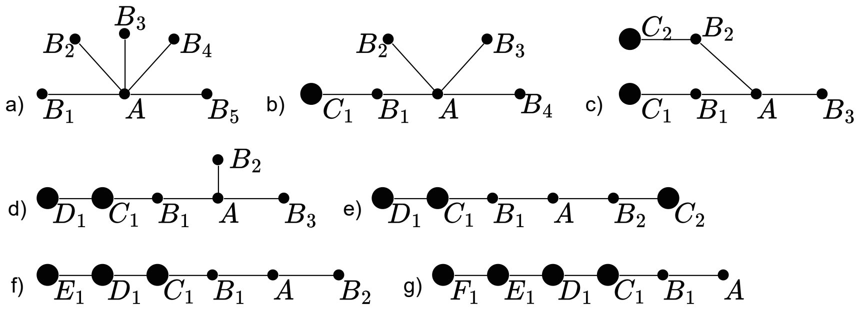


Figure 5.1: Dual graph: $(-K_S)^2 = 5$ for a general point

Then $\tau(A) = \frac{5}{2}$ and the Zariski decomposition of the divisor $\sigma^*(-K_S) - vE_P$ is given by:

a).
$$P(v) = \begin{cases} \sigma^*(-K_S) - vA \text{ if } v \in [0,2], \\ \sigma^*(-K_S) - vA - (v-2)(B_1+B_2+B_3+B_4+B_5) \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$
$$N(v) = \begin{cases} 0 \text{ if } v \in [0,2], \\ (v-2)(B_1+B_2+B_3+B_4+B_5) \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$

b).
$$P(v) = \begin{cases} \sigma^*(-K_S) - vA \text{ if } v \in [0,2], \\ \sigma^*(-K_S) - vA - (v-2)(2B_1+C_1+B_2+B_3+B_4) \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$
$$N(v) = \begin{cases} 0 \text{ if } v \in [0,2], \\ (v-2)(2B_1+C_1+B_2+B_3+B_4) \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$

c).
$$P(v) = \begin{cases} \sigma^*(-K_S) - vA \text{ if } v \in [0,2], \\ \sigma^*(-K_S) - vA - (v-2)(2B_1+C_1+2B_2+C_2+B_3) \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$
$$N(v) = \begin{cases} 0 \text{ if } v \in [0,2], \\ (v-2)(2B_1+C_1+2B_2+C_2+B_3) \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$

d).
$$P(v) = \begin{cases} \sigma^*(-K_S) - vA \text{ if } v \in [0,2], \\ \sigma^*(-K_S) - vA - (v-2)(3B_1+2C_1+D_1+B_2+B_3) \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$
$$N(v) = \begin{cases} 0 \text{ if } v \in [0,2], \\ (v-2)(3B_1+2C_1+D_1+B_2+B_3) \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$

e).
$$P(v) = \begin{cases} \sigma^*(-K_S) - vA \text{ if } v \in [0,2], \\ \sigma^*(-K_S) - vA - (v-2)(3B_1+2C_1+D_1+2B_2+C_2) \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$
$$N(v) = \begin{cases} 0 \text{ if } v \in [0,2], \\ (v-2)(3B_1+2C_1+D_1+2B_2+C_2) \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$

f).
$$P(v)=\begin{cases}\sigma^*(-K_S)-vA \text{ if } v\in[0,2],\\ \sigma^*(-K_S)-vA-(v-2)(4B_1+3C_1+2D_1+E_1+B_2) \text{ if } v\in\left[2,\frac{5}{2}\right].\end{cases}$$
$$N(v)=\begin{cases}0 \text{ if } v\in[0,2],\\ (v-2)(4B_1+3C_1+2D_1+E_1+B_2) \text{ if } v\in\left[2,\frac{5}{2}\right].\end{cases}$$

g).
$$P(v)=\begin{cases}\sigma^*(-K_S)-vA \text{ if } v\in[0,2],\\ \sigma^*(-K_S)-vA-(v-2)(5B_1+4C_1+3D_1+2E_1+F_1) \text{ if } v\in\left[2,\frac{5}{2}\right].\end{cases}$$
$$N(v)=\begin{cases}0 \text{ if } v\in[0,2],\\ (v-2)(5B_1+4C_1+3D_1+2E_1+F_1) \text{ if } v\in\left[2,\frac{5}{2}\right].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}5-v^2 \text{ if } v\in[0,2],\\ (2v-5)^2 \text{ if } v\in\left[2,\frac{5}{2}\right].\end{cases}\qquad P(v)\cdot A=\begin{cases}v \text{ if } v\in[0,2],\\ 2(5-2v) \text{ if } v\in\left[2,\frac{5}{2}\right].\end{cases}$$

In this case $\delta_P(S)\geq\frac{6}{5}$.

Proof. The Zariski Decomposition in part a). follows from

$$\sigma^*(-K_S)-vA\sim_{\mathbb{R}}\Big(\frac{5}{2}-v\Big)A+\frac{1}{2}\Big(B_1+B_2+B_3+B_4+B_5\Big).$$

A similar statement holds in other parts. We have $S_S(A)=\frac{3}{2}$. Thus, $\delta_P(S)\leq\frac{2}{3/2}=\frac{4}{3}$. Moreover,

$$h(v)\leq\begin{cases}\frac{v^2}{2} \text{ if } v\in[0,2],\\ 2(5-2v)(3v-5) \text{ if } v\in\left[2,\frac{5}{2}\right].\end{cases}$$

So $S(W^A_{\bullet,\bullet};O)\leq\frac{5}{6}$. We get that $\delta_P(S)\geq\frac{6}{5}$. $\square$

Now we consider a curve A on S. Small circles correspond to a (-1)-curves and large circles correspond to a (-2)-curves on dual graphs.

Lemma 5.1.2. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

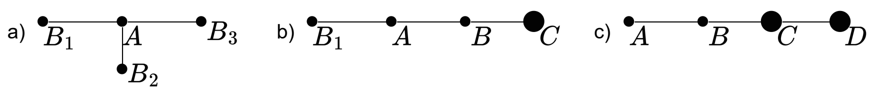


Figure 5.2: Dual graph: $(-K_S)^2=5$ and $\delta_P(S)=\frac{15}{13}$

Then $\tau(A)=2$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

a).
$$P(v)=\begin{cases}-K_S-vA \text{ if } v\in[0,1],\\ -K_S-vA-(v-1)(B_1+B_2+B_3) \text{ if } v\in[1,2].\end{cases}$$
$$N(v)=\begin{cases}0 \text{ if } v\in[0,1],\\ (v-1)(B_1+B_2+B_3) \text{ if } v\in[1,2].\end{cases}$$

b).
$$P(v)=\begin{cases}-K_S-vA \text{ if } v\in[0,1],\\ -K_S-vA-(v-1)(B_1+2B+C) \text{ if } v\in[1,2].\end{cases}$$
$$N(v)=\begin{cases}0 \text{ if } v\in[0,1],\\ (v-1)(B_1+2B+C) \text{ if } v\in[1,2].\end{cases}$$

c).
$$P(v)=\begin{cases}-K_S-vA \text{ if } v\in[0,1],\\ -K_S-vA-(v-1)(3B+2C+D) \text{ if } v\in[1,2].\end{cases}$$
$$N(v)=\begin{cases}0 \text{ if } v\in[0,1],\\ (v-1)(3B+2C+D) \text{ if } v\in[1,2].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}5-2v-v^2 \text{ if } v\in[0,1],\\ 2(2-v)^2 \text{ if } v\in[1,2].\end{cases}\quad P(v)\cdot A=\begin{cases}v+1 \text{ if } v\in[0,1],\\ 2(2-v) \text{ if } v\in[1,2].\end{cases}$$

In this case $\delta_P(S)=\frac{15}{13}$ if $P\in A\backslash B$.

Proof. The Zariski Decomposition in part a). follows from $-K_S-vA\sim_{\mathbb{R}}(2-v)A+B_1+B_2+B_3$. A similar statement holds in other parts. We have $S_S(A)=\frac{15}{13}$. Thus, $\delta_P(S)\le\frac{15}{13}$ for $P\in A$. Moreover for $P\in A\backslash B$,
$$h(v)\le\begin{cases}\frac{(v+1)^2}{2} \text{ if } v\in[0,1],\\ 2(2-v) \text{ if } v\in[1,2].\end{cases}$$
So $S(W^A_{\bullet,\bullet};P)\le\frac{15}{13}$. We get that $\delta_P(S)=\frac{15}{13}$ for $P\in A\backslash B$. □

Lemma 5.1.3. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

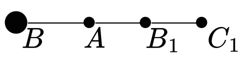


Figure 5.3: Dual graph: $(-K_S)^2=5$ and $\delta_P(S)=1$

Then $\tau(A)=2$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}B \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{2}B-(v-1)B_1 \text{ if } v\in[1,2].\end{cases}\qquad N(v)=\begin{cases}\frac{v}{2}B \text{ if } v\in[0,1],\\ \frac{v}{2}B+(v-1)B_1 \text{ if } v\in[1,2].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}5-2v-\frac{v^2}{2} \text{ if } v\in[0,1],\\ \frac{(2-v)(6-v)}{2} \text{ if } v\in[1,2].\end{cases}\qquad P(v)\cdot A=\begin{cases}1+\frac{v}{2} \text{ if } v\in[0,1],\\ 2-\frac{v}{2} \text{ if } v\in[1,2].\end{cases}$$

In this case $\delta_P(S)=1$ if $P\in A\backslash B$.

Proof. The Zariski Decomposition follows from $-K_S-vA\sim_{\mathbb{R}}(2-v)A+2B_1+C_1+B$. We have $S_S(A)=1$. Thus, $\delta_P(S)\leq 1$ for $P\in A$. Note that for $P\in A\backslash B$:

$$h(v)\leq\begin{cases}\frac{(v+2)^2}{8} \text{ if } v\in[0,1],\\ \frac{3(4-v)v}{8} \text{ if } v\in[1,2].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\leq\frac{13}{15}<1$. Thus, $\delta_P(S)=1$ if $P\in A\backslash B$. □

Lemma 5.1.4. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

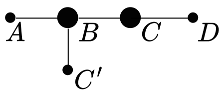


Figure 5.4: Dual graph: $(-K_S)^2=5$ and $\delta_P(S)=\frac{30}{31}$

Then $\tau(A)=2$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

$$P(v)=\begin{cases}-K_S-vA-\frac{v}{3}(2B+C) \text{ if } v\in\left[0,\frac{3}{2}\right],\\ -K_S-vA-(v-1)(2B+C)-(2v-3)C' \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{3}(2B+C) \text{ if } v\in\left[0,\frac{3}{2}\right],\\ (v-1)(2B+C)+(2v-3)C' \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}5-2v-\frac{v^2}{3} \text{ if } v\in\left[0,\frac{3}{2}\right],\\ (2-v)(4-v) \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}\qquad P(v)\cdot A=\begin{cases}1+\frac{v}{3} \text{ if } v\in\left[0,\frac{3}{2}\right],\\ 3-v \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

In this case $\delta_P(S)=\frac{30}{31}$ if $P\in A\backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + 3B + 2C + D + 2C'$. We have $S_S(A) = \frac{31}{30}$. Thus, $\delta_P(S) \leq \frac{30}{31}$ for $P \in A$. Moreover if $P \in A \backslash B$:

$$h(v) = \begin{cases} \frac{(v+3)^2}{18} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{(3-v)^2}{2} \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \leq \frac{19}{30} \leq \frac{31}{30}$. We get that $\delta_P(S) = \frac{30}{31}$ for $P \in A \backslash B$. □

Lemma 5.1.5. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

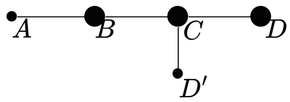


Figure 5.5: Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{15}{16}$

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = -K_S - vA - \frac{v}{4}(3B + 2C + D) \text{ and } N(v) = \frac{v}{4}(3B + 2C + D) \text{ if } v \in [0, 2].$$

Moreover,

$$(P(v))^2 = \frac{(2-v)(10+v)}{2} \text{ and } P(v) \cdot A = 1 + \frac{v}{4} \text{ if } v \in [0, 2].$$

In this case $\delta_P(S) = \frac{15}{16}$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + 3B + 4C + 2D + 3D'$. We have $S_S(A) = \frac{16}{15}$. Thus, $\delta_P(S) \leq \frac{15}{16}$ for $P \in A$. Moreover if $P \in A \backslash B$:

$$h(v) = \frac{(v+4)^2}{32} \text{ if } v \in [0, 2].$$

So $S(W^A_{\bullet,\bullet}; P) \leq \frac{19}{30} \leq \frac{16}{15}$. We get that $\delta_P(S) = \frac{15}{16}$ for $P \in A \backslash B$. □

Lemma 5.1.6. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

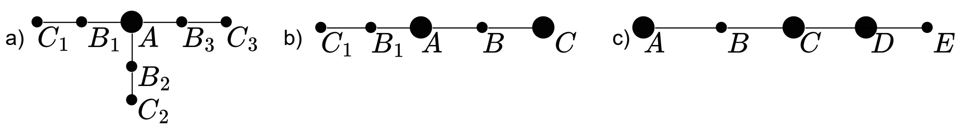


Figure 5.6: Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{15}{17}$ with $\tau(A) = 2$

Then $\tau(A)=2$ *and the Zariski Decomposition of the divisor* $-K_S-vA$ *is given by:*

a).
$$P(v)=\begin{cases}-K_S-vA \text{ if } v\in[0,1],\\ -K_S-vA-(v-1)(B_1+B_2+B_3) \text{ if } v\in[1,2].\end{cases}$$
$$N(v)=\begin{cases}0 \text{ if } v\in[0,1],\\ (v-1)(B_1+B_2+B_3) \text{ if } v\in[1,2].\end{cases}$$

b).
$$P(v)=\begin{cases}-K_S-vA \text{ if } v\in[0,1],\\ -K_S-vA-(v-1)(B_1+2B+C) \text{ if } v\in[1,2].\end{cases}$$
$$N(v)=\begin{cases}0 \text{ if } v\in[0,1],\\ (v-1)(B_1+2B+C) \text{ if } v\in[1,2].\end{cases}$$

c).
$$P(v)=\begin{cases}-K_S-vA \text{ if } v\in[0,1],\\ -K_S-vA-(v-1)(3B+2C+D) \text{ if } v\in[1,2].\end{cases}$$
$$N(v)=\begin{cases}0 \text{ if } v\in[0,1],\\ (v-1)(3B+2C+D) \text{ if } v\in[1,2].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}5-2v^2 \text{ if } v\in[0,1],\\ (2-v)(4-v) \text{ if } v\in[1,2].\end{cases}\qquad P(v)\cdot A=\begin{cases}2v \text{ if } v\in[0,1],\\ 3-v \text{ if } v\in[1,2].\end{cases}$$

In this case $\delta_P(S)=\frac{15}{17}$ *if* $P\in A\backslash B$.

Proof. The Zariski Decomposition in part a). follows from

$$-K_S-vA\sim_{\mathbb{R}}(2-v)A+\frac{1}{3}\Big(4B_1+4B_2+4B_3+C_1+C_2+C_3\Big).$$

A similar statement holds in other parts. We have $S_S(A)=\frac{17}{15}$. Thus, $\delta_P(S)\leq\frac{15}{17}$ for $P\in A$. Note that we have:

$$h(v)\leq\begin{cases}2v^2 \text{ if } v\in[0,1],\\ \frac{(3-v)(v+1)}{2} \text{ if } v\in[1,2].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\leq 1\leq\frac{17}{15}$. Thus, $\delta_P(S)=\frac{15}{17}$ if $P\in A\backslash B$. □

Lemma 5.1.7. *Suppose* P *belongs to a* (-1)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

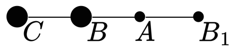


Figure 5.7: Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{15}{17}$ with $\tau(A) = 3$

Then $\tau(A) = 3$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2B+C) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{3}(2B+C) - (v-1)B_1 \text{ if } v \in [1,3]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2B+C) \text{ if } v \in [0,1], \\ \frac{v}{3}(2B+C) + (v-1)B_1 \text{ if } v \in [1,3]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 5 - 2v - \frac{v^2}{3} \text{ if } v \in [0,1], \\ \frac{2(3-v)^2}{3} \text{ if } v \in [1,3]. \end{cases} \quad P(v) \cdot A = \begin{cases} 1 + \frac{v}{3} \text{ if } v \in [0,1], \\ 2(1 - \frac{v}{3}) \text{ if } v \in [1,3]. \end{cases}$$

In this case $\delta_P(S) = \frac{15}{17}$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (3-v)A + 2B + C + 2B_1$. We have $S_S(A) = \frac{17}{15}$. Thus, $\delta_P(S) \leq \frac{15}{17}$ for $P \in A$. Moreover, for $P \in A \backslash B$:

$$h(v) \leq \begin{cases} \frac{(v+3)^2}{18} \text{ if } v \in [0,1], \\ \frac{4(3-v)}{9} \text{ if } v \in [1,3]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{13}{15} \leq \frac{17}{15}$. We get that $\delta_P(S) = \frac{15}{17}$ for $P \in A \backslash B$. □

Lemma 5.1.8. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

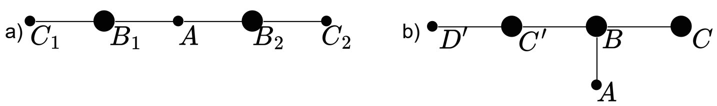


Figure 5.8: Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{15}{19}$ with $\tau(A) = 3$

Then $\tau(A) = 3$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}(B_1+B_2) \text{ if } v \in [0,2], \\ -K_S - vA - (v-1)(B_1+B_2) - (v-2)(C_1+C_2) \text{ if } v \in [2,3]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(B_1+B_2) \text{ if } v \in [0,2], \\ (v-1)(B_1+B_2) + (v-2)(C_1+C_2) \text{ if } v \in [2,3]. \end{cases}$$

b). $$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}(2B+C+C') \text{ if } v \in [0,2], \\ -K_S - vA - (v-1)(2B+C) - (2v-3)C' - (2v-4)D' \text{ if } v \in [2,3]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{2}(2B+C+C') \text{ if } v \in [0,2], \\ (v-1)(2B+C) + (2v-3)C' + (2v-4)D' \text{ if } v \in [2,3]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 5-2v \text{ if } v \in [0,2], \\ (3-v)^2 \text{ if } v \in [2,3]. \end{cases} \qquad P(v)\cdot A = \begin{cases} 1 \text{ if } v \in [0,2], \\ 3-v \text{ if } v \in [2,3]. \end{cases}$$

In this case $\delta_P(S) = \frac{15}{19}$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} (3-v)A + 2B_1 + C_1 + 2B_2 + C_2$. A similar statement holds in other parts. We have $S_S(A) = \frac{19}{15}$. Thus, $\delta_P(S) \leq \frac{15}{19}$ for $P \in A$. Moreover,

$$h(v) \leq \begin{cases} \frac{1+v}{2} \text{ if } v \in [0,2], \\ \frac{(3-v)(v+1)}{2} \text{ if } v \in [2,3]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{17}{15} \leq \frac{19}{15}$. We get that $\delta_P(S) = \frac{15}{19}$ for $P \in A$. $\square$

Lemma 5.1.9. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

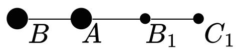


Figure 5.9: Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{15}{19}$ with $\tau(A) = 2$

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B - (v-1)B_1 \text{ if } v \in [1,2]. \end{cases} \qquad N(v) = \begin{cases} \frac{v}{2}B \text{ if } v \in [0,1], \\ \frac{v}{2}B + (v-1)B_1 \text{ if } v \in [1,2]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 5 - \frac{3v^2}{2} \text{ if } v \in [0,1], \\ \frac{(2-v)(6+v)}{2} \text{ if } v \in [1,2]. \end{cases} \qquad P(v)\cdot A = \begin{cases} \frac{3v}{2} \text{ if } v \in [0,1], \\ 1 + \frac{v}{2} \text{ if } v \in [1,2]. \end{cases}$$

In this case $\delta_P(S) = \frac{15}{19}$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + 3B_1 + 2C_1 + B$. We have $S_S(A) = \frac{19}{15}$. Thus, $\delta_P(S) \le \frac{15}{19}$ for $P \in A$. Moreover, if $P \in A \backslash B$ then:

$$h(v) \le \begin{cases} \frac{9v^2}{8} \text{ if } v \in [0,1], \\ \frac{(v+2)(5v-2)}{8} \text{ if } v \in [1,2]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \le \frac{17}{15} \le \frac{19}{15}$. We get that $\delta_P(S) = \frac{15}{19}$ for $P \in A \backslash B$. □

Lemma 5.1.10. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

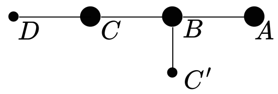


Figure 5.10: Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{10}{13}$

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2B+C) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ -K_S - vA - (v-1)(2B+C) - (2v-3)C' \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2B+C) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ (v-1)(2B+C) + (2v-3)C' \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 5 - \frac{4v^2}{3} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ 4(2-v) \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{4v}{3} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ 2 \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{10}{13}$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + 4B + 3C + 2D + 3C'$. We have $S_S(A) = \frac{13}{10}$. Thus, $\delta_P(S) \le \frac{10}{13}$ for $P \in A$. Moreover if $P \in A \backslash B$:

$$h(v) = \begin{cases} \frac{8v^2}{9} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ 2 \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \le \frac{4}{5} \le \frac{13}{10}$. We get that $\delta_P(S) = \frac{10}{13}$ for $P \in A \backslash B$. □

Lemma 5.1.11. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

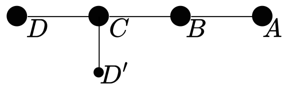


Figure 5.11: Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{3}{4}$

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = -K_S - vA - \frac{v}{4}(3B+2C+D) \text{ and } N(v) = \frac{v}{4}(3B+2C+D) \text{ if } v \in [0,2].$$

Moreover,

$$(P(v))^2 = \frac{5(2-v)(2+v)}{2} \text{ and } P(v)\cdot A = \frac{5v}{4} \text{ if } v \in [0,2].$$

In this case $\delta_P(S) = \frac{3}{4}$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + 4B + 6C + 3D + 5D'$. We have $S_S(A) = \frac{4}{3}$. Thus, $\delta_P(S) \le \frac{3}{4}$ for $P \in A$. Moreover if $P \in A\backslash B$ then $h(v) = \frac{25v^2}{32}$ if $v \in [0,2]$. So $S(W^A_{\bullet,\bullet};P) \le \frac{5}{6} \le \frac{4}{3}$. We get that $\delta_P(S) = \frac{3}{4}$ for $P \in A\backslash B$. □

Lemma 5.1.12. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

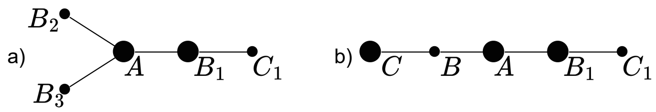


Figure 5.12: Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{5}{7}$

Then $\tau(A) = 3$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B_1 - (v-1)(B_2+B_3) \text{ if } v \in [1,2], \\ -K_S - vA - (v-1)(B_1+B_2+B_3) - (v-2)C_1 \text{ if } v \in [2,3]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ \frac{v}{2}B_1 + (v-1)(B_2+B_3) \text{ if } v \in [1,2], \\ (v-1)(B_1+B_2+B_3) + (v-2)C_1 \text{ if } v \in [2,3]. \end{cases}$$

b).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B_1 - (v-1)(2B_2+C_2) \text{ if } v \in [1,2], \\ -K_S - vA - (v-1)(B_1+2B_2+C_2) - (v-2)C_1 \text{ if } v \in [2,3]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ \frac{v}{2}B_1 + (v-1)(2B_2+C_2) \text{ if } v \in [1,2], \\ (v-1)(B_1+2B_2+C_2) + (v-2)C_1 \text{ if } v \in [2,3]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 5-\frac{3v^2}{2} \text{ if } v\in[0,1], \\ 7-4v+\frac{v^2}{2} \text{ if } v\in[1,2], \\ (3-v)^2 \text{ if } v\in[2,3]. \end{cases} \quad P(v)\cdot A = \begin{cases} \frac{3v}{2} \text{ if } v\in[0,1], \\ 2-\frac{v}{2} \text{ if } v\in[1,2], \\ 3-v \text{ if } v\in[2,3]. \end{cases}$$

In this case $\delta_P(S)=\frac{5}{7}$ *if* $P\in A\backslash B$.

Proof. The Zariski Decomposition in part a). follows from $-K_S-vA\sim_{\mathbb{R}}(3-v)A+2B_1+C_1+2B_2+2B_3$. A similar statement holds in other parts. We have $S_S(A)=\frac{7}{5}$. Thus, $\delta_P(S)\leq\frac{5}{7}$ for $P\in A$. Moreover, if $P\in A\cap B_1$ or if $P\in A\backslash B_1$ then:

$$h(v) = \begin{cases} \frac{15v^2}{8} \text{ if } v\in[0,1], \\ \frac{(4-v)(v+4)}{8} \text{ if } v\in[1,2], \\ \frac{(3-v)(v+1)}{2} \text{ if } v\in[2,3]. \end{cases} \quad \text{or } h(v)\leq \begin{cases} \frac{9v^2}{8} \text{ if } v\in[0,1], \\ \frac{3(4-v)v}{8} \text{ if } v\in[1,2], \\ \frac{(3-v)(v+1)}{2} \text{ if } v\in[2,3]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\leq\frac{19}{15}\leq\frac{7}{5}$ or $S(W^A_{\bullet,\bullet};P)\leq\frac{31}{30}\leq\frac{7}{5}$. We get that $\delta_P(S)=\frac{5}{7}$ for $P\in A$. □

Lemma 5.1.13. *Suppose* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

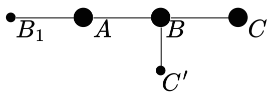


Figure 5.13: Dual graph: $(-K_S)^2=5$ and $\delta_P(S)=\frac{30}{43}$

Then $\tau(A)=3$ *and the Zariski Decomposition of the divisor* $-K_S-vA$ *is given by:*

$$P(v) = \begin{cases} -K_S-vA-\frac{v}{3}(2B+C) \text{ if } v\in[0,1], \\ -K_S-vA-\frac{v}{3}(2B+C)-(v-1)B_1 \text{ if } v\in\left[1,\frac{3}{2}\right], \\ -K_S-vA-(v-1)(2B+C+B_1)-(2v-3)C' \text{ if } v\in\left[\frac{3}{2},3\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2B+C) \text{ if } v\in[0,1], \\ \frac{v}{3}(2B+C)+(v-1)B_1 \text{ if } v\in\left[1,\frac{3}{2}\right], \\ (v-1)(2B+C+B_1)+(2v-3)C' \text{ if } v\in\left[\frac{3}{2},3\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 5-\frac{4v^2}{3} \text{ if } v\in[0,1], \\ 6-2v-\frac{v^2}{3} \text{ if } v\in\left[1,\frac{3}{2}\right], \\ (3-v)^2 \text{ if } v\in\left[\frac{3}{2},3\right]. \end{cases} \quad P(v)\cdot A = \begin{cases} \frac{4v}{3} \text{ if } v\in[0,1], \\ 1+\frac{v}{3} \text{ if } v\in\left[1,\frac{3}{2}\right], \\ 3-v \text{ if } v\in\left[\frac{3}{2},3\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{30}{43}$ *if* $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (3-v)A + 4B + 2C + 3C' + 2B_1$. We have $S_S(A) = \frac{43}{30}$. Thus, $\delta_P(S) \le \frac{30}{43}$ for $P \in A$. Note that if $P \in A\backslash B$

$$h(v) = \begin{cases} \frac{8v^2}{9} \text{ if } v \in [0,1], \\ \frac{(v+3)(7v-3)}{18} \text{ if } v \in \left[1, \frac{3}{2}\right], \\ \frac{(3-v)(v+1)}{2} \text{ if } v \in \left[\frac{3}{2}, 3\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \le \frac{16}{15} \le \frac{43}{30}$. Thus, $\delta_P(S) = \frac{30}{43}$ if $P \in A\backslash B$. □

Lemma 5.1.14. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

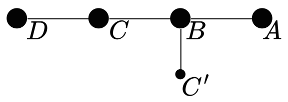


Figure 5.14: Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{9}{13}$

Then $\tau(A) = 3$ *and the Zariski Decomposition of the divisor* $-K_S - vA$ *is given by:*

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{4}(3B + 2C + D) \text{ if } v \in \left[0, \frac{4}{3}\right], \\ -K_S - vA - (v-1)(3B + 2C + D) - (3v-4)C' \text{ if } v \in \left[\frac{4}{3}, 3\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(3B + 2C + D) \text{ if } v \in \left[0, \frac{4}{3}\right], \\ (v-1)(3B + 2C + D) + (3v-4)C' \text{ if } v \in \left[\frac{4}{3}, 3\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} \frac{5(2-v)(2+v)}{2} \text{ if } v \in \left[0, \frac{4}{3}\right], \\ (3-v)^2 \text{ if } v \in \left[\frac{4}{3}, 3\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{5v}{4} \text{ if } v \in \left[0, \frac{4}{3}\right], \\ 3 - v \text{ if } v \in \left[\frac{4}{3}, 3\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{9}{13}$ *if* $P \in A\backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (3-v)A + 6B + 4C + 5C' + 2D$. We have $S_S(A) = \frac{13}{9}$. Thus, $\delta_P(S) \le \frac{9}{13}$ for $P \in A$. Moreover if $P \in A\backslash B$:

$$h(v) = \begin{cases} \frac{25}{32}v^2 \text{ if } v \in \left[0, \frac{4}{3}\right], \\ \frac{(3-v)^2}{2} \text{ if } v \in \left[\frac{4}{3}, 3\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \le \frac{5}{9} \le \frac{13}{9}$. We get that $\delta_P(S) = \frac{9}{13}$ for $P \in A\backslash B$. □

Lemma 5.1.15. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

Figure 5.15: Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{15}{23}$

Then $\tau(A) = 4$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2B_1 + C_1) - \frac{v}{2}B_2 \text{ if } v \in [0,3], \\ -K_S - vA - (v-1)B_1 - (v-2)C_1 - (v-3)D_1 - \frac{v}{2}B_2 \text{ if } v \in [3,4]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2B_1 + C_1) + \frac{v}{2}B_2 \text{ if } v \in [0,3], \\ (v-1)B_1 + (v-2)C_1 + (v-3)D_1 + \frac{v}{2}B_2 \text{ if } v \in [3,4]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 5 - 2v + \frac{v^2}{6} \text{ if } v \in [0,3], \\ \frac{(4-v)^2}{2} \text{ if } v \in [3,4]. \end{cases} \quad P(v) \cdot A = \begin{cases} 1 - \frac{v}{6} \text{ if } v \in [0,3], \\ 2 - \frac{v}{2} \text{ if } v \in [3,4]. \end{cases}$$

In this case $\delta_P(S) = \frac{15}{23}$ if $P \in A$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (4-v)A + 3B_1 + 2C_1 + D_1 + 2B_2$. We have $S_S(A) = \frac{23}{15}$. Thus, $\delta_P(S) \leq \frac{15}{23}$ for $P \in A$. Moreover, if $P \in A \backslash B_1$ or if $P \in A \cap B_1$:

$$h(v) \leq \begin{cases} \frac{(6-v)(5v+6)}{72} \text{ if } v \in [0,3], \\ \frac{(4-v)(v+4)}{8} \text{ if } v \in [3,4]. \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{(6-v)(7v+6)}{72} \text{ if } v \in [0,3], \\ \frac{3(4-v)v}{8} \text{ if } v \in [3,4]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{17}{15} \leq \frac{23}{15}$ or $S(W^A_{\bullet,\bullet};P) \leq \frac{7}{5} \leq \frac{23}{15}$. We get that $\delta_P(S) = \frac{15}{23}$ for $P \in A$. □

Lemma 5.1.16. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

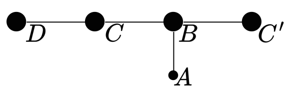


Figure 5.16: Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{3}{5}$

Then $\tau(A) = 5$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = -K_S - vA - \frac{v}{5}(2D + 4C + 6B + 3C') \text{ and } N(v) = \frac{v}{5}(2D + 4C + 6B + 3C') \text{ if } v \in [0,5].$$

Moreover,

$$(P(v))^2 = \frac{(5-v)^2}{2} \text{ and } P(v) \cdot A = 1 - \frac{v}{5} \text{ if } v \in [0,5].$$

In this case $\delta_P(S)=\frac{3}{5}$ *if* $P\in A\backslash B$.

Proof. The Zariski Decomposition follows from $-K_S-vA\sim_{\mathbb{R}}(5-v)A+3C'+6B+4C+2D$. We have $S_S(A)=\frac{5}{3}$. Thus, $\delta_P(S)\le\frac{3}{5}$ for $P\in A$. Moreover if $P\in A\backslash B$ then $h(v)=\frac{(5-v)^2}{50}$ if $v\in[0,5]$. So $S(W^A_{\bullet,\bullet};P)\le\frac{1}{3}\le\frac{5}{3}$. We get that $\delta_P(S)=\frac{3}{5}$ for $P\in A\backslash B$. □

Lemma 5.1.17. *Suppose* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

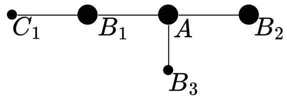


Figure 5.17: Dual graph: $(-K_S)^2=5$ and $\delta_P(S)=\frac{5}{9}$

Then $\tau(A)=4$ *and the Zariski Decomposition of the divisor* $-K_S-vA$ *is given by:*

$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}(B_1+B_2) \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{2}(B_1+B_2)-(v-1)B_3 \text{ if } v\in[1,2],\\ -K_S-vA-(v-1)(B_1+B_3)-\frac{v}{2}B_2-(v-2)C_1 \text{ if } v\in[2,4].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{2}(B_1+B_2) \text{ if } v\in[0,1],\\ \frac{v}{2}(B_1+B_2)+(v-1)B_3 \text{ if } v\in[1,2],\\ (v-1)(B_1+B_3)+\frac{v}{2}B_2+(v-2)C_1 \text{ if } v\in[2,4].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}5-v^2 \text{ if } v\in[0,1],\\ 2(3-v) \text{ if } v\in[1,2],\\ \frac{(4-v)^2}{2} \text{ if } v\in[2,4].\end{cases}\quad P(v)\cdot A=\begin{cases}v \text{ if } v\in[0,1],\\ 1 \text{ if } v\in[1,2],\\ 2-\frac{v}{2} \text{ if } v\in[2,4].\end{cases}$$

In this case $\delta_P(S)=\frac{5}{9}$ *if* $P\in A$.

Proof. The Zariski Decomposition follows from $-K_S-vA\sim_{\mathbb{R}}(4-v)A+3B_1+2C_1+2B_2+3B_3$. We have $S_S(A)=\frac{9}{5}$. Thus, $\delta_P(S)\le\frac{5}{9}$ for $P\in A$. Moreover, if $P\in A\backslash(B_2\cup B_3)$ or if $P\in A\cap B_2$ or if $P\in A\cap B_3$:

$$h(v)\le\begin{cases}v^2 \text{ if } v\in[0,1],\\ \frac{(v+1)}{2} \text{ if } v\in[1,2],\\ \frac{3(4-v)v}{8} \text{ if } v\in[2,4].\end{cases}\quad \text{or } h(v)=\begin{cases}v^2 \text{ if } v\in[0,1],\\ \frac{(v+1)}{2} \text{ if } v\in[1,2],\\ \frac{(4-v)(v+4)}{8} \text{ if } v\in[2,4].\end{cases}\quad \text{or } h(v)=\begin{cases}\frac{v^2}{2} \text{ if } v\in[0,1],\\ v-\frac{1}{2} \text{ if } v\in[1,2],\\ \frac{3(4-v)v}{8} \text{ if } v\in[2,4].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\le\frac{43}{30}\le\frac{9}{5}$ or $S(W^A_{\bullet,\bullet};P)\le\frac{13}{10}\le\frac{9}{5}$ or $S(W^A_{\bullet,\bullet};P)\le\frac{19}{15}\le\frac{9}{5}$. We get that $\delta_P(S)=\frac{5}{9}$ for $P\in A$. □

Lemma 5.1.18. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

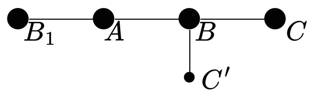


Figure 5.18: Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{6}{11}$

Then $\tau(A) = 4$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B_1 - \frac{v}{3}(2B+C) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ -K_S - vA - \frac{v}{2}B_1 - (v-1)(2B+C) - (2v-3)C' \text{ if } v \in \left[\frac{3}{2}, 4\right]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{2}B_1 + \frac{v}{3}(2B+C) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{v}{2}B_1 + (v-1)(2B+C) + (2v-3)C' \text{ if } v \in \left[\frac{3}{2}, 4\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 5 - \frac{5v^2}{2} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{(4-v)^2}{2} \text{ if } v \in \left[\frac{3}{2}, 4\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{5v}{6} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ 2 - \frac{v}{2} \text{ if } v \in \left[\frac{3}{2}, 4\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{6}{11}$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (4-v)A + 2B_1 + 6B + 3C + 5C'$. We have $S_S(A) = \frac{11}{6}$. Thus, $\delta_P(S) \le \frac{6}{11}$ for $P \in A$. Moreover, if $P \in A \backslash B$ then:

$$h(v) \le \begin{cases} \frac{55v^2}{72} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{(4-v)(v+4)}{8} \text{ if } v \in \left[\frac{3}{2}, 4\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \le \frac{4}{3} \le \frac{11}{6}$. We get that $\delta_P(S) = \frac{6}{11}$ for $P \in A \backslash B$. $\square$

Lemma 5.1.19. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

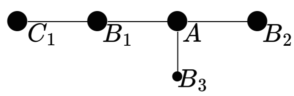


Figure 5.19: Dual graph: $(-K_S)^2 = 5$ and $\delta_P(S) = \frac{3}{7}$

Then $\tau(A) = 6$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2B_1 + C_1) - \frac{v}{2}B_2 \text{ if } v \in [0, 1], \\ -K_S - vA - \frac{v}{3}(2B_1 + C_1) - \frac{v}{2}B_2 - (v-1)B_3 \text{ if } v \in [1, 6]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2B_1+C_1)+\frac{v}{2}B_2 \text{ if } v \in [0,1], \\ \frac{v}{3}(2B_1+C_1)+\frac{v}{2}B_2+(v-1)B_3 \text{ if } v \in [1,6]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 5-5\frac{v^2}{2} \text{ if } v \in [0,1], \\ \frac{(6-v)^2}{2} \text{ if } v \in [1,6]. \end{cases} \quad P(v)\cdot A = \begin{cases} \frac{5v}{6} \text{ if } v \in [0,1], \\ 1-\frac{v}{6} \text{ if } v \in [1,6]. \end{cases}$$

In this case $\delta_P(S) = \frac{3}{7}$ *if* $P \in A$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (6-v)A+4B_1+2C_1+3B_2+5B_3$. We have $S_S(A) = \frac{7}{3}$. Thus, $\delta_P(S) \leq \frac{3}{7}$ for $P \in A$. Moreover, if $P \in A \cap B_3$ or if $P \in A \backslash B_3$ then:

$$h(v) = \begin{cases} \frac{25v^2}{72} \text{ if } v \in [0,1], \\ \frac{(6-v)(11v-6)}{72} \text{ if } v \in [1,6]. \end{cases} \quad \text{or } h(v) = \begin{cases} \frac{65v^2}{72} \text{ if } v \in [0,1], \\ \frac{(6-v)(7v+6)}{72} \text{ if } v \in [1,6]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{5}{3} \leq \frac{7}{3}$ or $S(W^A_{\bullet,\bullet};P) \leq \frac{11}{6} \leq \frac{7}{3}$. We get that $\delta_P(S) = \frac{3}{7}$ for $P \in A$. □

5.2 Finding δ-invariants for degree 5

Let X be a singular del Pezzo surface of degree 5 with and S be a minimal resolution of X. Then there are several possible cases:

I. X has an $\mathbb{A}_1$ singularity and contains 7 lines. In this case, we let E be the exceptional divisor, L_{123}, L_i, L_i' for $i \in \{1,2,3\}$ be the lines on S,

II. X has two $\mathbb{A}_1$ singularities and contains 5 lines. In this case, we let E_i for $i \in \{1,2\}$ be the exceptional divisors, L_{12}, L_i, L_i' for $i \in \{1,2\}$ be the lines on S,

III. X has an $\mathbb{A}_2$ singularity and contains 4 lines. In this case, we let E_i for $i \in \{1,2\}$ be the exceptional divisors, L_i, L_i' for $i \in \{1,2\}$ be the lines on S,

IV. X has $\mathbb{A}_2$ and $\mathbb{A}_1$ singularities and contains 3 lines. In this case, we let E_i for $i \in \{1,2,3\}$ be the exceptional divisors, L_{13} L_2, L_2' be the lines on S,

V. X has an $\mathbb{A}_3$ singularity and contains 2 lines. In this case, we let E_i for $i \in \{1,2,3\}$ be the exceptional divisors, L_1 and L_2 be the lines on S,

VI. X has an $\mathbb{A}_4$ singularity and contains 1 line. In this case, we let E_i for $i \in \{1,2,3,4\}$ be the exceptional divisors, L_3 be the line on S.

such that the dual graph of the (-1)-curves and (-2)-curves on S is given on the picture below:

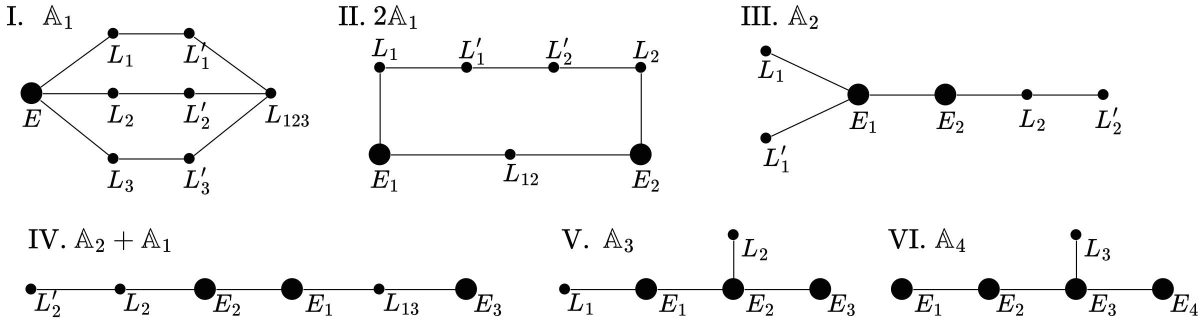


Figure 5.20: Du Val del Pezzo surfaces with $(-K_S)^2 = 5$

One has:

I. $\delta(X) = \frac{15}{17}$ since depending on the position of point $P \in S$ we have

P	E	$(L_1 \cup L_2 \cup L_3)\backslash E$	$(L_1' \cup L_2' \cup L_3' \cup L_{123})\backslash(L_1 \cup L_2 \cup L_3)$	o/w
$\delta_P(S)$	$\frac{15}{17}$	1	$\frac{15}{13}$	$\geq \frac{6}{5}$

Table 5.2: Local δ-invariants: $(-K_S)^2 = 5$ and $\mathbb{A}_1$ singularity

II. $\delta(X) = \frac{15}{19}$ since depending on the position of point $P \in S$ we have

P	L_{12}	$(E_1 \cup E_2)\backslash L_{12}$	$(L_1 \cup L_2)\backslash(E_1 \cup E_2)$	$(L_1' \cup L_2')\backslash(L_1 \cup L_2)$	o/w
$\delta_P(S)$	$\frac{15}{19}$	$\frac{15}{17}$	1	$\frac{15}{13}$	$\geq \frac{6}{5}$

Table 5.3: Local δ-invariants: $(-K_S)^2 = 5$ and $2\mathbb{A}_1$ singularities

III. $\delta(X) = \frac{5}{7}$ since depending on the position of point $P \in S$ we have

P	E_1	$E_2\backslash E_1$	$L_2\backslash E_2$	$(L_1 \cup L_1')\backslash E_1$	$L_2'\backslash L_2$	o/w
$\delta_P(S)$	$\frac{5}{7}$	$\frac{15}{19}$	$\frac{15}{17}$	$\frac{30}{31}$	$\frac{15}{13}$	$\geq \frac{6}{5}$

Table 5.4: Local δ-invariants: $(-K_S)^2 = 5$ and $\mathbb{A}_2$ singularity

IV. $\delta(X) = \frac{15}{23}$ since depending on the position of point $P \in S$ we have

P	L_{13}	$E_1\backslash L_{13}$	$E_2\backslash E_1$	$(E_3 \cup L_2)\backslash(L_{13} \cup E_2)$	$L_2'\backslash L_2$	o/w
$\delta_P(S)$	$\frac{15}{23}$	$\frac{5}{7}$	$\frac{15}{19}$	$\frac{15}{17}$	$\frac{15}{13}$	$\geq \frac{6}{5}$

Table 5.5: Local δ-invariants: $(-K_S)^2 = 5$ and $\mathbb{A}_2\mathbb{A}_1$ singularities

V. $\delta(X) = \frac{5}{9}$ since depending on the position of point $P \in S$ we have

P	E_2	$E_1\backslash E_2$	$E_3\backslash E_2$	$L_2\backslash E_2$	$L_1\backslash E_1$	o/w
$\delta_P(S)$	$\frac{5}{9}$	$\frac{30}{43}$	$\frac{10}{13}$	$\frac{15}{19}$	$\frac{15}{16}$	$\geq \frac{6}{5}$

Table 5.6: Local δ-invariants: $(-K_S)^2 = 5$ and $\mathbb{A}_3$ singularity

VI. $\delta(X) = \frac{3}{7}$ since depending on the position of point $P \in S$ we have

P	E_3	$E_2 \backslash E_3$	$L_3 \backslash E_3$	$E_4 \backslash E_3$	$E_1 \backslash E_2$	o/w
$\delta_P(S)$	$\frac{3}{7}$	$\frac{6}{11}$	$\frac{3}{5}$	$\frac{9}{13}$	$\frac{3}{4}$	$\geq \frac{6}{5}$

Table 5.7: Local δ-invariants: $(-K_S)^2 = 5$ and $\mathbb{A}_4$ singularity

Proof. We prove each case separately using lemmas from the previous section.

I. If $P \in E$ the assertion follows from Lemma 5.1.6. If $P \in (L_1 \cup L_2 \cup L_3) \backslash E$ the assertion follows from Lemma 5.1.3. If $P \in L_{123} \backslash (L_1' \cup L_2' \cup L_3')$ the assertion follows from Lemma 5.1.2 [a).]. If $P \in (L_1' \cup L_2' \cup L_3') \backslash (L_1 \cup L_2 \cup L_3)$ the assertion follows from Lemma 5.1.2 [b).]. If P is a general point the assertion follows from Lemma 5.1.1.

II. If $P \in L_{12}$ the assertion follows from Lemma 5.1.8. If $P \in (E_1 \cup E_2) \backslash L_{12}$ the assertion follows from Lemma 5.1.6 [b).]. If $P \in (L_1 \cup L_2) \backslash (E_1 \cup E_2)$ the assertion follows from Lemma 5.1.3. If $P \in (L_1' \cup L_2') \backslash (L_1 \cup L_2)$ the assertion follows from Lemma 5.1.2 [b).]. If P is a general point the assertion follows from Lemma 5.1.1.

III. If $P \in E_1$ the assertion follows from Lemma 5.1.12 [a).]. If $P \in E_2 \backslash E_1$ the assertion follows from Lemma 5.1.9. If $P \in L_2 \backslash E_2$ the assertion follows from Lemma 5.1.7. If $P \in (L_1 \cup L_1') \backslash E_1$ the assertion follows from Lemma 5.1.4. If $P \in L_2' \backslash L_2$ the assertion follows from Lemma 5.1.2 [c).]. If P is a general point the assertion follows from Lemma 5.1.1.

IV. If $P \in L_{13}$ the assertion follows from Lemma 5.1.15. If $P \in E_1 \backslash L_{13}$ the assertion follows from Lemma 5.1.12 [b).]. If $P \in E_2 \backslash E_1$ the assertion follows from Lemma 5.1.9. If $P \in L_2 \backslash E_2$ the assertion follows from Lemma 5.1.7. If $P \in E_3 \backslash L_{13}$ the assertion follows from Lemma 5.1.6. If $P \in L_2' \backslash L_2$ the assertion follows from Lemma 5.1.2 [c).]. If P is a general point the assertion follows from Lemma 5.1.1.

V. If $P \in E_2$ the assertion follows from Lemma 5.1.17. If $P \in E_1 \backslash E_2$ the assertion follows from Lemma 5.1.13. If $P \in E_3 \backslash E_2$ the assertion follows from Lemma 5.1.10. If $P \in L_2 \backslash E_2$ the assertion follows from Lemma 5.1.8 [b)]. If $P \in L_1 \backslash E_1$ the assertion follows from Lemma 5.1.5. If P is a general point the assertion follows from Lemma 5.1.1.

VI. If $P \in E_3$ the assertion follows from Lemma 5.1.19. If $P \in E_2 \backslash E_3$ the assertion follows from Lemma 5.1.18. If $P \in L_3 \backslash E_3$ the assertion follows from Lemma 5.1.16. If $P \in E_4 \backslash E_3$ the assertion follows from Lemma 5.1.14. If $P \in E_1 \backslash E_2$ the assertion follows from Lemma 5.1.11. If P is a general point the assertion follows from Lemma 5.1.1.

□

Chapter 6

Du Val del Pezzo Surfaces of Degree 4

In (Araujo et al., 2023, Lemma 2.12) it was proven that $\delta(X) = \frac{4}{3}$ when X is a smooth del Pezzo surface of degree 4. In this chapter, we compute δ-invariants of singular Du Val del Pezzo surfaces of degree 4.

MAIN THEOREM Let X be a singular Du Val del Pezzo surface of degree 4. Then the δ-invariant of X is uniquely determined by the degree of X, the number of lines on X, and the type of singularities on X which is given in the following table:

K_X^2	# lines	Sing(X)	$\delta(X)$
4	12	$\mathbb{A}_1$	1
4	9	$2\mathbb{A}_1$	1
4	8	$2\mathbb{A}_1$	1
4	6	$3\mathbb{A}_1$	1
4	4	$4\mathbb{A}_1$	1

K_X^2	# lines	Sing(X)	$\delta(X)$
4	8	$\mathbb{A}_2$	$\frac{6}{7}$
4	6	$\mathbb{A}_2+\mathbb{A}_1$	$\frac{6}{7}$
4	4	$\mathbb{A}_2+2\mathbb{A}_1$	$\frac{6}{7}$
4	5	$\mathbb{A}_3$	$\frac{2}{3}$
4	4	$\mathbb{A}_3$	$\frac{3}{4}$

K_X^2	# lines	Sing(X)	$\delta(X)$
4	3	$\mathbb{A}_3+\mathbb{A}_1$	$\frac{3}{4}$
4	2	$\mathbb{A}_3+2\mathbb{A}_1$	$\frac{3}{4}$
4	3	$\mathbb{A}_4$	$\frac{6}{11}$
4	2	$\mathbb{D}_4$	$\frac{1}{2}$
4	1	$\mathbb{D}_5$	$\frac{3}{8}$

Table 6.2: δ-invariants of Du Val del Pezzo surfaces of degree 4

6.1 General results for degree 4

Let X be a del Pezzo surface of degree 4 with at most Du Val singularities, S be a minimal resolution of X and P is a point on S. Then:

Lemma 6.1.1. *If P is a general point on S. Consider the blowup $\sigma : \widetilde{S} \to S$ of S at P with the exceptional divisor E_P. Such that $\tau(E_P) = 2$ and the Zariski decomposition of the divisor $\sigma^*(-K_S) - vE_P$ is given by: $P(v) = \sigma^*(-K_S) - vE_P$ and $N(v) = 0$ if $v \in [0,2]$. and $P(v)^2 = (2-v)(2+v)P(v)\cdot E_P = v$ if $v \in [0,2]$. In this case $\delta_P(S) = \frac{3}{2}$.*

Proof. The Zariski Decomposition follows from $\sigma^*(-K_S) - vE_P \sim_{\mathbb{R}} (2-v)E_P + Q + L$ where Q and L are (-1)-curves which are strict transforms of a conic and a line on $\mathbb{P}^2$ respectively. We have $S_S(E_P) = \frac{4}{3}$. Thus, $\delta_P(S) \leq \frac{2}{4/3} = \frac{3}{2}$ Moreover if $O \in E_P$ we have $h(v) = \frac{v^2}{2}$ if $v \in [0,2]$. So $S(W^{E_P}_{\bullet,\bullet}; O) \leq \frac{2}{3}$ Thus, $\delta_P(S) = \frac{3}{2}$. □

Lemma 6.1.2. *Suppose $P \in Q$ where Q is a (-1)-curve on S such that it does not intersect (-2)-curves. Let L_P be a (1)-curve which intersects with Q at this point with multiplicity 2 (one can see that such curve exists by considering explicit models). Consider the blowup of S at P with the exceptional divisor E_P. After that we blow up the intersection of strict transforms Q, L_P, and E_P with the exceptional divisor E, and contract a (-2)-curve. We call the resulting surface $\overline{S}$. Let $\overline{L}_P$, $\overline{Q}$, $\overline{E}$, and $\overline{P}$ be the images of strict transforms of L_P, Q, E and E_P respectively. Note that $\overline{P}$ is a point. We denote this weighted blowup by $\sigma : \overline{S} \to S$.*

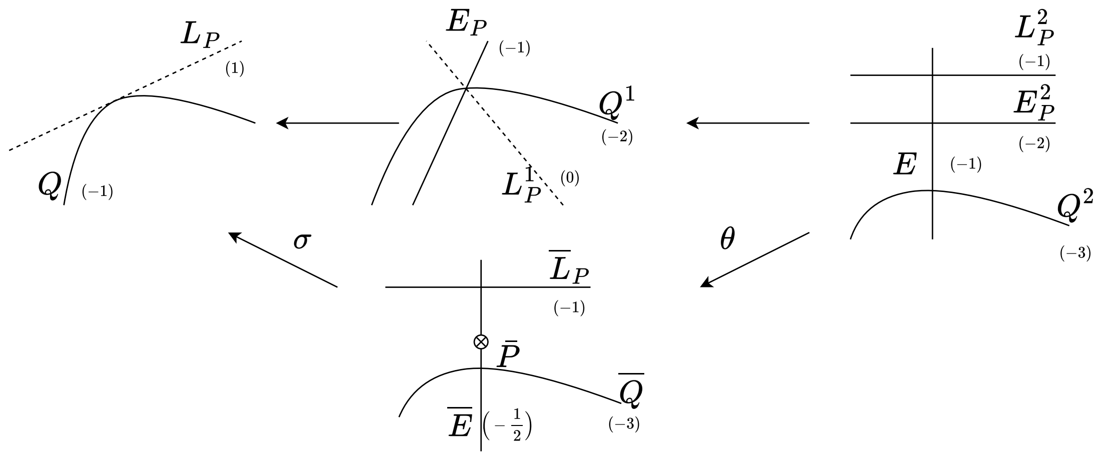


Figure 6.1: $(-K_S)^2 = 4$, a general point on a (-1)-curve

The intersections are given by

	$\overline{Q}$	$\overline{E}$	$\overline{L}_P$
$\overline{Q}$	-3	1	0
$\overline{E}$	1	$-\frac{1}{2}$	1
$\overline{L}_P$	0	1	-1

Table 6.4: $(-K_S)^2 = 4$, a general point on a (-1)-curve

Then $\tau(\overline{E}) = 4$ and the Zariski decomposition of the divisor $\sigma^*(-K_S) - v\overline{E}$ is given by:

$$P(v) = \begin{cases} \sigma^*(-K_S) - v\overline{E} \text{ if } v \in [0,1], \\ \sigma^*(-K_S) - v\overline{E} - \frac{(v-1)}{3}\overline{Q} \text{ if } v \in [1,3], \\ \sigma^*(-K_S) - v\overline{E} - \frac{(v-1)}{3}\overline{Q} - (v-3)\overline{L}_P \text{ if } v \in [3,4]. \end{cases}$$

$$N(v) = \begin{cases} 0 \text{ if } v \in [0,1], \\ \frac{(v-1)}{3}\overline{Q} \text{ if } v \in [1,3], \\ \frac{(v-1)}{3}\overline{Q} + (v-3)\overline{L}_P \text{ if } v \in [3,4]. \end{cases}$$

	E_P	$\mathscr{A}_1$	$\mathscr{A}_2$	$\mathscr{A}_3$
E_P	-1	1	1	1
$\mathscr{A}_1$	1	-2	0	0
$\mathscr{A}_2$	1	0	-2	0
$\mathscr{A}_3$	1	0	0	-1

Table 6.5: $(-K_S)^2 = 4$, the intersection of two (-1)-curves

Moreover

$$P(v)^2 = \begin{cases} 4 - \frac{v^2}{2} \text{ if } v \in [0,1], \\ \frac{26-4v-v^2}{6} \text{ if } v \in [1,3], \\ \frac{5(4-v)^2}{6} \text{ if } v \in [3,4]. \end{cases} \quad P(v)\cdot \overline{E} = \begin{cases} \frac{v}{2} \text{ if } v \in [0,1], \\ \frac{2+v}{6} \text{ if } v \in [1,3], \\ \frac{5(4-v)}{6} \text{ if } v \in [3,4]. \end{cases}$$

In this case $\delta_P(S) = \frac{18}{13}$.

Proof. The Zariski Decomposition follows from $\sigma^*(-K_S) - v\overline{E} \sim \overline{L}_P + \overline{Q} + (4-v)\overline{E}$. We have $S_S(\overline{E}) = \frac{13}{6}$. Thus, since $A_S(\overline{E}) = 3$ then $\delta_P(S) \leq \frac{3}{13/6} = \frac{18}{13}$. Moreover, if $O \in \overline{E}\backslash(\overline{Q} \cup \overline{L}_P)$ or $O \in \overline{E} \cap \overline{Q}$ or if $O \in \overline{E} \cap \overline{L}_P$:

$$h(v) \leq \begin{cases} \frac{v^2}{8} \text{ if } v \in [0,1], \\ \frac{(v+2)^2}{72} \text{ if } v \in [1,3], \\ \frac{25(v-4)^2}{72} \text{ if } v \in [3,4]. \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{v^2}{8} \text{ if } v \in [0,1], \\ \frac{(v+2)(5v-2)}{72} \text{ if } v \in [1,3], \\ \frac{5(4-v)(16-v)}{72} \text{ if } v \in [3,4]. \end{cases}$$

$$\text{or } h(v) \leq \begin{cases} \frac{v^2}{8} \text{ if } v \in [0,1], \\ \frac{(v+2)^2}{72} \text{ if } v \in [1,3], \\ \frac{5(4-v)(7v-16)}{72} \text{ if } v \in [3,4]. \end{cases}$$

Thus if $O \in \overline{E}\backslash(\overline{Q} \cup \overline{L}_P)$ then $S(W^{\overline{E}}_{\bullet,\bullet};O) = \frac{11}{36}$, if $O \in \overline{E} \cap \overline{Q}$ then $S(W^{\overline{E}}_{\bullet,\bullet};O) = \frac{17}{24}$, if $O \in E_P \cap L_P$ then $S(W^{\overline{E}}_{\bullet,\bullet};O) = \frac{3}{8}$. Now, to get a lower bound for $\delta_P(S)$, we use Corollary 2.2.2 that gives

$$\delta_P(S) \geqslant \min\left\{\frac{18}{13}, \inf_{O \in \overline{E}} \frac{A_{\overline{E},\Delta_{\overline{E}}}(O)}{S(W^{\overline{E}}_{\bullet,\bullet};O)}\right\},$$

where $\Delta_{\overline{E}} = \frac{1}{2}\theta(\overline{E})$. So $\delta_P(S) \geq \min\left\{\frac{18}{13}, \frac{36}{11}, \frac{36}{22}, \frac{24}{17}, \frac{8}{3}\right\} = \frac{18}{13}$ thus $\delta_P(S) = \frac{18}{13}$. □

Lemma 6.1.3. *Suppose A_1 and A_2 are (-1)-curves on S which intersect at point P. Consider the blowup $\sigma : \widetilde{S} \to S$ of S at P with the exceptional divisor E_P. Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ are strict transforms of A_1 and A_2 on $\widetilde{S}$ and $\mathscr{A}_3$ is another (-1)-curve on S such that the intersections are given by Then $\tau(E_P) = 3$ and the Zariski decomposition of the divisor $\sigma^*(-K_S) - vE_P$ is* *given by:*

$$P(v) = \begin{cases} \sigma^*(-K_S) - vE_P \text{ if } v \in [0,1], \\ \sigma^*(-K_S) - vE_P - \frac{(v-1)}{2}(\mathscr{A}_1 + \mathscr{A}_2) \text{ if } v \in [1,2], \\ \sigma^*(-K_S) - vE_P - \frac{(v-1)}{2}(\mathscr{A}_1 + \mathscr{A}_2) - (v-2)\mathscr{A}_3 \text{ if } v \in [2,3]. \end{cases}$$

$$N(v) = \begin{cases} 0 \text{ if } v \in [0,1], \\ \frac{(v-1)}{2}(\mathscr{A}_1 + \mathscr{A}_2) \text{ if } v \in [1,2], \\ \frac{(v-1)}{2}(\mathscr{A}_1 + \mathscr{A}_2) + (v-2)\mathscr{A}_3 \text{ if } v \in [2,3]. \end{cases}$$

with

$$P(v)^2 = \begin{cases} (2-v)(2+v) \text{ if } v \in [0,1], \\ 5-2v \text{ if } v \in [1,2], \\ (3-v)^2 \text{ if } v \in [2,3]. \end{cases} \qquad P(v) \cdot E_P = \begin{cases} v \text{ if } v \in [0,1], \\ 1 \text{ if } v \in [1,2], \\ 3-v \text{ if } v \in [2,3]. \end{cases}$$

In this case $\delta_P(S) = \frac{4}{3}$.

Proof. The Zariski Decomposition follows from $\sigma^*(-K_S) - vE_P \sim_{\mathbb{R}} (3-v)E_P + \mathscr{A}_1 + \mathscr{A}_2 + \mathscr{A}_3$. We have $S_S(E_P) = \frac{3}{2}$. Thus $\delta_P(S) \le \frac{2}{3/2} = \frac{4}{3}$. Moreover, if $O \in E_P \backslash \mathscr{A}_3$ or if $O \in E_P \cap \mathscr{A}_3$ then:

$$h(v) \le \begin{cases} \frac{v^2}{2} \text{ if } v \in [0,1], \\ \frac{v}{2} \text{ if } v \in [1,2], \\ 3-v \text{ if } v \in [2,3]. \end{cases} \quad \text{or } h(v) \le \begin{cases} \frac{v^2}{2} \text{ if } v \in [0,1], \\ \frac{1}{2} \text{ if } v \in [1,2], \\ \frac{(3-v)(v-1)}{2} \text{ if } v \in [2,3]. \end{cases}$$

So $S(W^{E_P}_{\bullet,\bullet}; O) \le \frac{17}{24} \le \frac{3}{4}$ or $S(W^{E_P}_{\bullet,\bullet}; O) \le \frac{1}{2} \le \frac{3}{4}$ Thus, $\delta_P(S) = \frac{4}{3}$ for $A_1 \cap A_2$. □

Now we consider a curve A on S. Small circles correspond to a (-1)-curves and large circles correspond to a (-2)-curves on dual graphs.

Lemma 6.1.4. *If P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

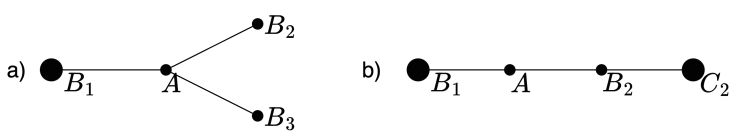


Figure 6.2: Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{6}{5}$

Then $\tau(A)=2$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

a).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}B_1 \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{2}B_1-(v-1)(B_2+B_3) \text{ if } v\in[1,2].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{2}B_1 \text{ if } v\in[0,1],\\ \frac{v}{2}B_1+(v-1)(B_2+B_3) \text{ if } v\in[1,2].\end{cases}$$

b).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}B_1 \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{2}B_1-(v-1)(2B_2+C_2) \text{ if } v\in[1,2].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{2}B_1 \text{ if } v\in[0,1],\\ \frac{v}{2}B_1+(v-1)(2B_2+C_2) \text{ if } v\in[1,2].\end{cases}$$

Moreover:

$$(P(v))^2=\begin{cases}4-2v-\frac{v^2}{2} \text{ if } v\in[0,1],\\ \frac{3(2-v)^2}{2} \text{ if } v\in[1,2].\end{cases}\quad P(v)\cdot A=\begin{cases}1+\frac{v}{2} \text{ if } v\in[0,1],\\ 3-\frac{3v}{2} \text{ if } v\in[1,2].\end{cases}$$

In this case $\delta_P(S)=\frac{6}{5}$ if $P\in A\backslash B_1$.

Proof. The Zariski Decomposition in part a). follows from $-K_S-vA\sim_{\mathbb{R}}(2-v)A+B_1+B_2+B_3$. A similar statement holds in other parts. We have $S_S(A)=\frac{5}{6}$. Thus, $\delta_P(S)\le\frac{6}{5}$ for $P\in A$. Moreover,
$$h(v)=\begin{cases}\frac{(v+2)^2}{8} \text{ if } v\in[0,1],\\ \frac{3(2-v)(5v-2)}{8} \text{ if } v\in[1,2].\end{cases}$$
So $S(W^A_{\bullet,\bullet};P)\le\frac{5}{6}$. We get that $\delta_P(S)=\frac{6}{5}$ for $P\in A\backslash B_1$. □

Lemma 6.1.5. *If P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

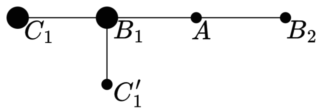


Figure 6.3: Dual graph: $(-K_S)^2=4$ and $\delta_P(S)=\frac{8}{7}$

Then $\tau(A)=2$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

$$P(v)=\begin{cases}-K_S-vA-\frac{v}{3}(2B_1+C_1) \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{3}(2B_1+C_1)-(v-1)B_2 \text{ if } v\in\left[1,\frac{3}{2}\right],\\ -K_S-vA-(v-1)(2B_1+C_1+B_2)-(2v-3)C_1' \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{3}(2B_1+C_1) \text{ if } v\in[0,1],\\ \frac{v}{3}(2B_1+C_1)+(v-1)B_2 \text{ if } v\in\left[1,\frac{3}{2}\right],\\ (v-1)(2B_1+C_1+B_2)+(2v-3)C_1' \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}4-2v-\frac{v^2}{3} \text{ if } v\in[0,1],\\ 5-4v+\frac{2v^2}{3} \text{ if } v\in\left[1,\frac{3}{2}\right],\\ 2(v-2)^2 \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}\quad P(v)\cdot A=\begin{cases}1+\frac{v}{3} \text{ if } v\in[0,1],\\ 2(1-\frac{v}{3}) \text{ if } v\in\left[1,\frac{3}{2}\right],\\ 4-2v \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

In this case $\delta_P(S)=\frac{8}{7}$ *if* $P\in A\backslash B_1$.

Proof. The Zariski Decomposition follows from $-K_S-vA\sim_{\mathbb{R}}(2-v)A+2B_1+C_1+C_1'+B_2$. We have $S_S(A)=\frac{7}{8}$. Thus, $\delta_P(S)\le\frac{8}{7}$ for $P\in A$. Note that for $P\in A\backslash B_1$ we have:

$$h(v)=\begin{cases}\frac{(v+3)^2}{18} \text{ if } v\in[0,1],\\ \frac{4v(3-v)}{9} \text{ if } v\in\left[1,\frac{3}{2}\right],\\ 2(2-v) \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\le\frac{17}{24}\le\frac{7}{8}$. Thus, $\delta_P(S)=\frac{8}{7}$ if $P\in A\backslash B_1$. □

Lemma 6.1.6. *If P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

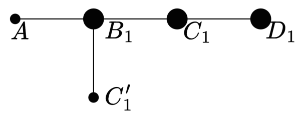


Figure 6.4: Dual graph: $(-K_S)^2=4$ and $\delta_P(S)=\frac{9}{8}$

Then $\tau(A)=2$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

$$P(v)=\begin{cases}-K_S-vA-\frac{v}{4}(3B_1+2C_1+D_1) \text{ if } v\in\left[0,\frac{4}{3}\right],\\ -K_S-vA-(v-1)(3B_1+2C_1+D_1)-(3v-4)C_1' \text{ if } v\in\left[\frac{4}{3},2\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{4}(3B_1+2C_1+D_1) \text{ if } v\in\left[0,\frac{4}{3}\right],\\ (v-1)(3B_1+2C_1+D_1)+(3v-4)C_1' \text{ if } v\in\left[\frac{4}{3},2\right].\end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 4-2v-\frac{v^2}{4} \text{ if } v \in \left[0,\frac{4}{3}\right], \\ 2(v-2)^2 \text{ if } v \in \left[\frac{4}{3},2\right]. \end{cases} \quad P(v)\cdot A = \begin{cases} 1+\frac{v}{4} \text{ if } v \in \left[0,\frac{4}{3}\right], \\ 2(2-v) \text{ if } v \in \left[\frac{4}{3},2\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{9}{8}$ if $P \in A \backslash B_1$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + 3B_1 + 2C_1 + D_1 + 2C_1'$. Thus, $S_S(A) = \frac{8}{9}$. Thus, $\delta_P(S) \leq \frac{9}{8}$ for $P \in A$. Note that we have that if $P \in A \backslash B_1$ then

$$h(v) = \begin{cases} \frac{(v+4)^2}{32} \text{ if } v \in \left[0,\frac{4}{3}\right], \\ 2(2-v)^2 \text{ if } v \in \left[\frac{4}{3},2\right]. \end{cases}$$

So for $S(W^A_{\bullet,\bullet};P) \leq \frac{5}{9} \leq \frac{8}{9}$. Thus, $\delta_P(S) = \frac{9}{8}$ if $P \in A \backslash B_1$. □

Lemma 6.1.7. *If P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

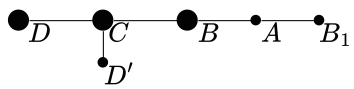


Figure 6.5: Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{12}{11}$

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{4}(3B+2C+D) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{4}(3B+2C+D) - (v-1)B_1 \text{ if } v \in [1,2]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(3B+2C+D) \text{ if } v \in [0,1], \\ \frac{v}{4}(3B+2C+D) + (v-1)B_1 \text{ if } v \in [1,2]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 4-2v-\frac{v^2}{4} \text{ if } v \in [0,1], \\ \frac{(2-v)(10-3v)}{4} \text{ if } v \in [1,2]. \end{cases} \quad P(v)\cdot A = \begin{cases} 2v \text{ if } \in [0,1] \\ 2-\frac{3v}{4} \text{ if } v \in [1,2]. \end{cases}$$

In this case $\delta_P(S) = \frac{12}{11}$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + 2B + 2C + D + D' + B_1$. We have $S_S(A) = \frac{11}{12}$. Thus, $\delta_P(S) \leq \frac{12}{11}$ for $P \in A$. Note that for $P \in A \backslash B$ we have:

$$h(v) \leq \begin{cases} \frac{(v+4)^2}{32} \text{ if } v \in [0,1], \\ \frac{5(8-3v)v}{32} \text{ if } v \in [1,2]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{17}{24} \leq \frac{11}{12}$. Thus, $\delta_P(S) = \frac{12}{11}$ if $P \in A \backslash B$. □

Lemma 6.1.8. *If P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

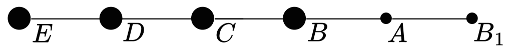


Figure 6.6: Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{24}{23}$

Then $\tau(A) = \frac{5}{2}$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{5}(4B + 3C + 2D + E) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{5}(4B + 3C + 2D + E) - (v-1)B_1 \text{ if } v \in \left[1, \frac{5}{2}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(4B + 3C + 2D + E) \text{ if } v \in [0,1], \\ \frac{v}{5}(4B + 3C + 2D + E) + (v-1)B_1 \text{ if } v \in \left[1, \frac{5}{2}\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 4 - 2v - \frac{v^2}{5} \text{ if } v \in [0,1], \\ \frac{(2v-5)^2}{5} \text{ if } v \in \left[1, \frac{5}{2}\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} 1 + \frac{v}{5} \text{ if } v \in [0,1], \\ 2 - \frac{4v}{5} \text{ if } v \in \left[1, \frac{5}{2}\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{24}{23}$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from

$$-K_S - vA \sim_{\mathbb{R}} \left(\frac{5}{2} - v\right)A + \frac{1}{2}\Big(4B + 3C + 2D + E + 3B_1\Big).$$

We have $S_S(A) = \frac{23}{24}$. Thus, $\delta_P(S) \leq \frac{24}{23}$ for $P \in A \backslash B$. Note that we have:

$$h(v) \leq \begin{cases} \frac{(v+5)^2}{50} \text{ if } v \in [0,1], \\ \frac{6v(5-2v)}{25} \text{ if } v \in \left[1, \frac{5}{2}\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{73}{120} \leq \frac{23}{24}$. Thus, $\delta_P(S) = \frac{24}{23}$ if $P \in A \backslash B$. □

Lemma 6.1.9. *If P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

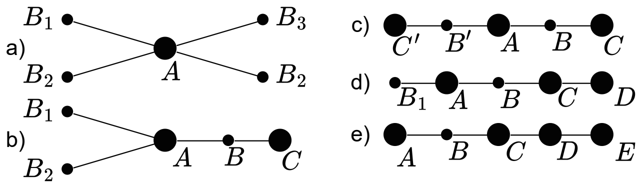


Figure 6.7: Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = 1$ with $-K_S - vA$ nef on $[0,1]$

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).
$$P(v) = \begin{cases} -K_S - vA \text{ if } v \in [0,1], \\ -K_S - vA - (v-1)(B_1 + B_2 + B_3 + B_4) \text{ if } v \in [1,2]. \end{cases}$$
$$N(v) = \begin{cases} 0 \text{ if } v \in [0,1], \\ (v-1)(B_1 + B_2 + B_3 + B_4) \text{ if } v \in [1,2]. \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vA \text{ if } v \in [0,1], \\ -K_S - vA - (v-1)(2B + C + B_1 + B_2) \text{ if } v \in [1,2]. \end{cases}$$
$$N(v) = \begin{cases} 0 \text{ if } v \in [0,1], \\ (v-1)(2B + C + B_1 + B_2) \text{ if } v \in [1,2]. \end{cases}$$

c).
$$P(v) = \begin{cases} -K_S - vA \text{ if } v \in [0,1], \\ -K_S - vA - (v-1)(2B + C + 2B' + C') \text{ if } v \in [1,2]. \end{cases}$$
$$N(v) = \begin{cases} 0 \text{ if } v \in [0,1], \\ (v-1)(2B + C + 2B' + C') \text{ if } v \in [1,2]. \end{cases}$$

d).
$$P(v) = \begin{cases} -K_S - vA \text{ if } v \in [0,1], \\ -K_S - vA - (v-1)(3B + 2C + D + B_1) \text{ if } v \in [1,2] \end{cases}$$
$$N(v) = \begin{cases} 0 \text{ if } v \in [0,1] \\ (v-1)(3B + 2C + D + B_1) \text{ if } v \in [1,2]. \end{cases}$$

e).
$$P(v) = \begin{cases} -K_S - vA \text{ if } v \in [0,1], \\ -K_S - vA - (v-1)(4B + 3C + 2D + E) \text{ if } v \in [1,2]. \end{cases}$$
$$N(v) = \begin{cases} 0 \text{ if } v \in [0,1], \\ (v-1)(4B + 3C + 2D + E) \text{ if } v \in [1,2]. \end{cases}$$

Moreover:

$$(P(v))^2 = \begin{cases} 4 - 2v^2 \text{ if } v \in [0,1], \\ 2(2-v)^2 \text{ if } v \in [1,2]. \end{cases} \quad P(v) \cdot A = \begin{cases} 2v \text{ if } v \in [0,1], \\ 2(2-v) \text{ if } v \in [1,2]. \end{cases}$$

In this case $\delta_P(S) = 1$ *if* $P \in A \backslash B$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + B_1 + B_2 + B_3 + B_4$. A similar statement holds in other parts. Consider a point P described above. We have $S_S(A) = 1$. Thus, $\delta_P(S) \leq 1$ for $P \in A$. Note that we have:

$$h(v) \leq \begin{cases} 2v^2 \text{ if } v \in [0,1], \\ 2v(2-v) \text{ if } v \in [1,2]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \leq 1$. Thus, $\delta_P(S) = 1$. □

Lemma 6.1.10. *If* P *belongs to a* (-1)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

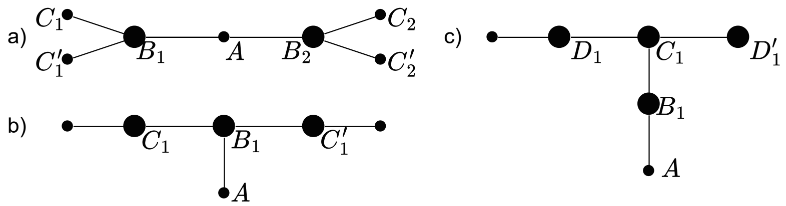


Figure 6.8: Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = 1$ with $-K_S - vA$ nef on $[0,2]$

Then $\tau(A) = 2$ *and the Zariski Decomposition of the divisor* $-K_S - vA$ *is given by:*

a).
$$P(v) = -K_S - vA - \frac{v}{2}(B_1 + B_2) \text{ if } v \in [0,2].$$
$$N(v) = \frac{v}{2}(B_1 + B_2) \text{ if } v \in [0,2].$$

b).
$$P(v) = -K_S - vA - \frac{v}{2}(2B_1 + C_1 + C_1') \text{ if } v \in [0,2].$$
$$N(v) = \frac{v}{2}(2B_1 + C_1 + C_1') \text{ if } v \in [0,2].$$

c).
$$P(v) = -K_S - vA - \frac{v}{2}(2B_1 + 2C_1 + D_1 + D_1') \text{ if } v \in [0,2].$$
$$N(v) = \frac{v}{2}(2B_1 + 2C_1 + D_1 + D_1') \text{ if } v \in [0,2].$$

Moreover,

$$(P(v))^2 = 4 - 2v \text{ and } P(v) \cdot A = 1 \text{ if } v \in [0,2].$$

In this case $\delta_P(S) = 1$ *if* $P \in A \backslash (B_1 \cup B_2)$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + \frac{1}{2}\Big(3B_1 + C_1 + C_1' + 3B_2 + C_2 + C_2'\Big)$. A similar statement holds in other parts. We have $S_S(A) = 1$. Thus, $\delta_P(S) \leq 1$ for $P \in A$. Moreover, if $P \in A \backslash (B_1 \cup B_2)$ we have $h(v) = 1/2$ if $v \in [0,2]$. So $S(W^A_{\bullet,\bullet}; P) \leq \frac{1}{2} \leq 1$. We get that $\delta_P(S) = 1$ for $P \in A \backslash (B_1 \cup B_2)$. □

Lemma 6.1.11. *If P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

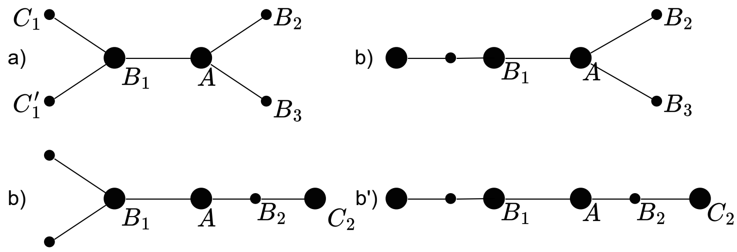


Figure 6.9: Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{6}{7}$ with $\tau(A) = 2$

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B_1 - (v-1)(B_2+B_3) \text{ if } v \in [1,2]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ \frac{v}{2}B_1 + (v-1)(B_2+B_3) \text{ if } v \in [1,2]. \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B_1 - (v-1)(2B_2+C_2) \text{ if } v \in [1,2]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ \frac{v}{2}B_1 + (v-1)(2B_2+C_2) \text{ if } v \in [1,2]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 4 - \frac{3v^2}{2} \text{ if } v \in [0,1], \\ \frac{(v-2)(v-6)}{2} \text{ if } v \in [1,2]. \end{cases} \qquad P(v) \cdot A = \begin{cases} 2v \text{ if } \frac{3v}{2} \in [0,1] \\ 2 - \frac{v}{2} \text{ if } v \in [1,2]. \end{cases}$$

In this case $\delta_P(S) = \frac{6}{7}$ if $P \in A$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + 2B_1 + C_1 + C_1' + B_2 + B_3$. A similar statement holds in other parts. We have $S_S(A) = \frac{7}{6}$. Thus, $\delta_P(S) \leq \frac{6}{7}$ for $P \in A$. Note that we have if $P \in A \backslash B_1$ or if $P \in A \cap B_1$:

$$h(v) \leq \begin{cases} \frac{9v^2}{8} \text{ if } v \in [0,1], \\ \frac{(4-v)(7v-4)}{8} \text{ if } v \in [1,2]. \end{cases} \quad \text{or} \quad \begin{cases} \frac{15v^2}{8} \text{ if } v \in [0,1], \\ \frac{(4-v)(4+v)}{8} \text{ if } v \in [1,2]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{7}{6}$. Thus, $\delta_P(S) = \frac{6}{7}$ if $P \in A$. □

Lemma 6.1.12. *If P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

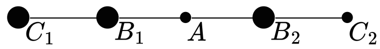


Figure 6.10: Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{6}{7}$ with $\tau(A) = 3$

Then $\tau(A) = 3$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2B_1 + C_1) - \frac{v}{2}B_2 \text{ if } v \in [0,2], \\ -K_S - vA - \frac{v}{3}(2B_1 + C_1) - (v-1)B_2 - (v-2)C_2 \text{ if } v \in [2,3]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2B_1 + C_1) + \frac{v}{2}B_2 \text{ if } v \in [0,2], \\ \frac{v}{3}(2B_1 + C_1) + (v-1)B_2 + (v-2)C_2 \text{ if } v \in [2,3]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 4 - 2v + \frac{v^2}{6} \text{ if } v \in [0,2], \\ \frac{2(v-3)^2}{3} \text{ if } v \in [2,3]. \end{cases} \quad P(v) \cdot A = \begin{cases} 2v \text{ if } 1 - \frac{v}{6} \in [0,2], \\ 2(1 - \frac{v}{3}) \text{ if } v \in [2,3]. \end{cases}$$

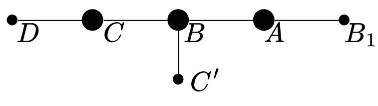
In this case $\delta_P(S) = \frac{6}{7}$ if $P \in A \backslash B_1$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (3-v)A + 2B_1 + C_1 + 2B_2 + C_2$. We have $S_S(A) = \frac{7}{6}$. Thus, $\delta_P(S) \leq \frac{6}{7}$ for $P \in A$. Note that for $P \in A \backslash B_1$ we have:

$$h(v) \leq \begin{cases} \frac{(6-v)(5v+6)}{72} \text{ if } v \in [0,2], \\ \frac{4v(3-v)}{9} \text{ if } v \in [2,3]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \leq 1 \leq \frac{7}{6}$. Thus, $\delta_P(S) = \frac{6}{7}$ if $P \in A \backslash B_1$. □

Lemma 6.1.13. *If P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*



Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2B + C) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{3}(2B + C) - (v-1)B_1 \text{ if } v \in \left[1, \frac{3}{2}\right], \\ -K_S - vA - (v-1)(2B + C + B_1) - (2v-3)C' \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

$$N(v)=\begin{cases}\frac{v}{3}(2B+C) \text{ if } v\in[0,1],\\ \frac{v}{3}(2B+C)+(v-1)B_1 \text{ if } v\in\left[1,\frac{3}{2}\right],\\ (v-1)(2B+C+B_1)+(2v-3)C' \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}4-\frac{4v^2}{3} \text{ if } v\in[0,1],\\ 5-2v-\frac{v^2}{3} \text{ if } v\in\left[1,\frac{3}{2}\right],\\ (2-v)(4-v) \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}\qquad P(v)\cdot A=\begin{cases}\frac{4v}{3} \text{ if } v\in[0,1],\\ 1+\frac{v}{3} \text{ if } v\in\left[1,\frac{3}{2}\right],\\ 3-v \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

In this case $\delta_P(S)=\frac{24}{29}$ *if* $P\in A\backslash B$.

Proof. The Zariski Decomposition follows from $-K_S-vA\sim_{\mathbb{R}}(2-v)A+3B+2C+D+2C'+B_1$. We have $S_S(A)=\frac{29}{24}$. Thus, $\delta_P(S)\le\frac{24}{29}$ for $P\in A$. Note that for $P\in A\backslash B$ we have:

$$h(v)=\begin{cases}\frac{8v^2}{9} \text{ if } v\in[0,1],\\ \frac{(v+3)(7v-3)}{18} \text{ if } v\in\left[1,\frac{3}{2}\right],\\ \frac{(3-v)(v+1)}{2} \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\le\frac{11}{12}\le\frac{29}{24}$. Thus, $\delta_P(S)=\frac{24}{29}$ if $P\in A\backslash B$. □

Lemma 6.1.14. *If* P *belongs to a* (-1)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

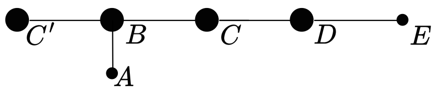


Figure 6.11: Dual graph: $(-K_S)^2=4$ and $\delta_P(S)=\frac{24}{29}$

Then $\tau(A)=3$ *and the Zariski Decomposition of the divisor* $-K_S-vA$ *is given by:*

$$P(v)=\begin{cases}-K_S-vA-\frac{v}{5}(3C'+6B+4C+2D) \text{ if } v\in\left[0,\frac{5}{2}\right],\\ -K_S-vA-(v-1)(2B+C')-(2v-3)C-(2v-4)D-(2v-5)E \text{ if } v\in\left[\frac{5}{2},3\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{5}(3C'+6B+4C+2D) \text{ if } v\in\left[0,\frac{5}{2}\right],\\ (v-1)(2B+C')+(2v-3)C+(2v-4)D+(2v-5)E \text{ if } v\in\left[\frac{5}{2},3\right].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}4-2v+\frac{v^2}{5} \text{ if } v\in\left[0,\frac{5}{2}\right],\\ (3-v)^2 \text{ if } v\in\left[\frac{5}{2},3\right].\end{cases}\qquad P(v)\cdot A=\begin{cases}1-\frac{v}{5} \text{ if } v\in\left[0,\frac{5}{2}\right],\\ 3-v \text{ if } v\in\left[\frac{5}{2},3\right].\end{cases}$$

In this case $\delta_P(S)=\frac{24}{29}$ *if* $P\in A\backslash B$.

Proof. The Zariski Decomposition follows from $-K_S-vA\sim_{\mathbb{R}}(3-v)A+2C'+4B+3C+2D+E$. We have $S_S(A)=\frac{29}{24}$. Thus, $\delta_P(S)\le\frac{24}{29}$ for $P\in A$. Note that if $P\in A\backslash B$ then

$$h(v)=\begin{cases}\frac{(5-v)^2}{50} \text{ if } v\in\left[0,\frac{5}{2}\right],\\ \frac{(3-v)^2}{2} \text{ if } v\in\left[\frac{5}{2},3\right].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\le\frac{3}{8}\le\frac{29}{24}$. Thus, $\delta_P(S)=\frac{24}{29}$ if $P\in A\backslash B$. □

Lemma 6.1.15. *If P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

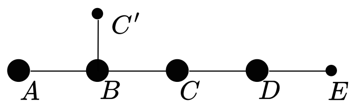


Figure 6.12: Dual graph: $(-K_S)^2=4$ and $\delta_P(S)=\frac{9}{11}$

Then $\tau(A)=2$ *and the Zariski Decomposition of the divisor* $-K_S-vA$ *is given by:*

$$P(v)=\begin{cases}-K_S-vA-\frac{v}{4}(3B+2C+D) \text{ if } v\in\left[0,\frac{4}{3}\right],\\ -K_S-vA-(v-1)(3B+2C+D)-(3v-4)C' \text{ if } v\in\left[\frac{4}{3},2\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{4}(3B+2C+D) \text{ if } v\in\left[0,\frac{4}{3}\right],\\ (v-1)(3B+2C+D)+(3v-4)C' \text{ if } v\in\left[\frac{4}{3},2\right].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}4-\frac{5v^2}{4} \text{ if } v\in\left[0,\frac{4}{3}\right],\\ (2-v)(4-v) \text{ if } v\in\left[\frac{4}{3},2\right].\end{cases}\qquad P(v)\cdot A=\begin{cases}\frac{5v}{4} \text{ if } v\in\left[0,\frac{4}{3}\right],\\ 3-v \text{ if } v\in\left[\frac{4}{3},2\right].\end{cases}$$

In this case $\delta_P(S)=\frac{9}{11}$ *if* $P\in A\backslash B$.

Proof. The Zariski Decomposition follows from $-K_S-vA\sim_{\mathbb{R}}(2-v)A+2C'+3B+3C+2D+E$. We have $S_S(A)=\frac{11}{9}$. Thus, $\delta_P(S)\le\frac{9}{11}$ for $P\in A$. Note that we have that if $P\in A\backslash B$ then

$$h(v)\le\begin{cases}\frac{25v^2}{32} \text{ if } v\in\left[0,\frac{4}{3}\right],\\ \frac{(3-v)^2}{2} \text{ if } v\in\left[\frac{4}{3},2\right].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\le\frac{11}{18}\le\frac{11}{9}$. Thus, $\delta_P(S)=\frac{9}{11}$ if $P\in A\backslash B$. □

Lemma 6.1.16. *If P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

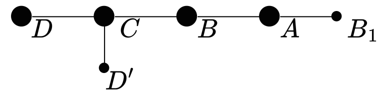


Figure 6.13: Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{4}{5}$

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{4}(3B+2C+D) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{4}(3B+2C+D) - (v-1)B_1 \text{ if } v \in [1,2]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(3B+2C+D) \text{ if } v \in [0,1], \\ \frac{v}{4}(3B+2C+D) + (v-1)B_1 \text{ if } v \in [1,2]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 4 - \frac{5v^2}{4} \text{ if } v \in [0,1], \\ \frac{(v+10)(2-v)}{4} \text{ if } v \in [1,2]. \end{cases} \quad P(v)\cdot A = \begin{cases} \frac{5v}{4} \text{ if } v \in [0,1], \\ 1 + \frac{v}{4} \text{ if } v \in [1,2]. \end{cases}$$

In this case $\delta_P(S) = \frac{4}{5}$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + 3B + 4C + 2D + 3D' + B_1$. We have $S_S(A) = \frac{5}{4}$. Thus, $\delta_P(S) \leq \frac{4}{5}$ for $P \in A$. Note that for $P \in A \backslash B$ we have:

$$h(v) \leq \begin{cases} \frac{25v^2}{32} \text{ if } v \in [0,1], \\ \frac{(v+4)(9v-4)}{32} \text{ if } v \in [1,2]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{23}{24} \leq \frac{5}{4}$. Thus, $\delta_P(S) = \frac{4}{5}$ if $P \in A \backslash B$. $\square$

Lemma 6.1.17. *If P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

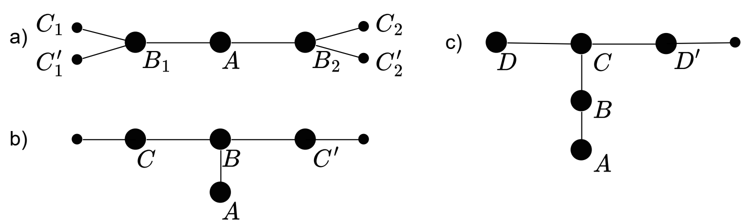


Figure 6.14: Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{3}{4}$ with $\tau(A) = 2$

Then $\tau(A)=2$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

a). $$P(v)=-K_S-vA-\frac{v}{2}(B_1+B_2) \text{ and } N(v)=\frac{v}{2}(B_1+B_2) \text{ if } v\in[0,2].$$

b). $$P(v)=-K_S-vA-\frac{v}{2}(2B+C+C') \text{ and } N(v)=\frac{v}{2}(2B+C+C') \text{ if } v\in[0,2].$$

c). $$P(v)=-K_S-vA-\frac{v}{2}(2B+2C+D+D') \text{ and } N(v)=\frac{v}{2}(2B+2C+D+D') \text{ if } v\in[0,2].$$

Moreover,

$$(P(v))^2=(2-v)(v+2) \text{ and } P(v)\cdot A=v \text{ if } v\in[0,2].$$

In this case $\delta_P(S)=\frac{3}{4}$ if $P\in A\backslash B$.

Proof. The Zariski Decomposition in part a). follows from $-K_S-vA\sim_{\mathbb{R}}(2-v)A+2B_1+C_1+C_1'+2B_2+C_2+C_2'$. A similar statement holds in other parts. We have $S_S(A)=\frac{4}{3}$. Thus, $\delta_P(S)\leq\frac{3}{4}$ for $P\in A$. Moreover $h(v)\leq v^2$ if $v\in[0,2]$. So $S(W^A_{\bullet,\bullet};P)\leq\frac{4}{3}$. We get that $\delta_P(S)=\frac{3}{4}$ $P\in A\backslash B$. □

Lemma 6.1.18. *If P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

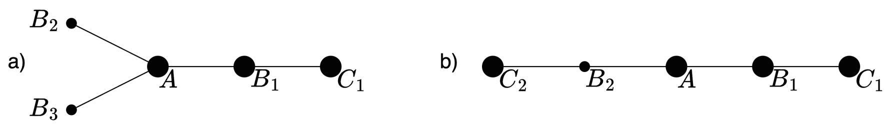


Figure 6.15: Dual graph: $(-K_S)^2=4$ and $\delta_P(S)=\frac{3}{4}$ with $\tau(A)=3$

Then $\tau(A)=3$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

a). $$P(v)=\begin{cases}-K_S-vA-\frac{v}{3}(2B_1+C_1) \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{3}(2B_1+C_1)-(v-1)(B_2+B_3) \text{ if } v\in[1,3].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{3}(2B_1+C_1) \text{ if } v\in[0,1],\\ \frac{v}{3}(2B_1+C_1)+(v-1)(B_2+B_3) \text{ if } v\in[1,3].\end{cases}$$

b). $$P(v)=\begin{cases}-K_S-vA-\frac{v}{3}(2B_1+C_1) \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{3}(2B_1+C_1)-(v-1)(2B_2+C_2) \text{ if } v\in[1,3].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{3}(2B_1+C_1) \text{ if } v\in[0,1],\\ \frac{v}{3}(2B_1+C_1)+(v-1)(2B_2+C_2) \text{ if } v\in[1,3].\end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 4-\frac{4v^2}{3} \text{ if } v\in[0,1], \\ \frac{2(3-v)^2}{3} \text{ if } v\in[1,3]. \end{cases} \quad P(v)\cdot A = \begin{cases} \frac{4v}{3} \text{ if } v\in[0,1], \\ 2(1-\frac{v}{3}) \text{ if } v\in[1,3]. \end{cases}$$

In this case $\delta_P(S)=\frac{3}{4}$ *if* $P\in A\backslash B_1$.

Proof. The Zariski Decomposition in part a). follows from $-K_S-vA\sim_{\mathbb{R}}(3-v)A+2B_1+C_1+2B_2+2B_3$. A similar statement holds in other parts. $S_S(A)=\frac{4}{3}$. Thus, $\delta_P(S)\leq\frac{3}{4}$ for $P\in A$. Note that for $P\in A\backslash B_1$ we have:

$$h(v)\leq \begin{cases} \frac{8v^2}{9} \text{ if } v\in[0,1], \\ \frac{2(3-v)(5v-3)}{9} \text{ if } v\in[1,3]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\leq\frac{4}{3}$. Thus, $\delta_P(S)=\frac{3}{4}$ if $P\in A\backslash B_1$. □

Lemma 6.1.19. *If* P *belongs to a* (-1)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

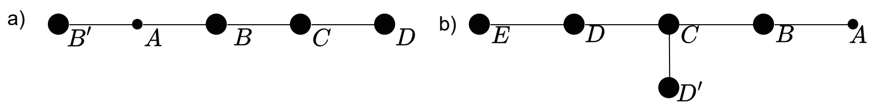


Figure 6.16: Dual graph: $(-K_S)^2=4$ and $\delta_P(S)=\frac{3}{4}$ with $\tau(A)=4$

Then $\tau(A)=4$ *and the Zariski Decomposition of the divisor* $-K_S-vA$ *is given by:*

a).
$$P(v)=-K_S-vA-\frac{v}{4}(2B'+3B+2C+D) \text{ if } v\in[0,4].$$
$$N(v)=\frac{v}{4}(2B'+3B+2C+D) \text{ if } v\in[0,4].$$

b).
$$P(v)=-K_S-vA-\frac{v}{4}(5B+6C+4D+2E+3D') \text{ if } v\in[0,4].$$
$$N(v)=\frac{v}{4}(5B+6C+4D+2E+3D') \text{ if } v\in[0,4].$$

Moreover,

$$(P(v))^2=\frac{(v-4)^2}{4} P(v)\cdot A=1-\frac{v}{4} \text{ if } v\in[0,4].$$

In this case $\delta_P(S)=\frac{3}{4}$ *if* $P\in A$.

Proof. The Zariski Decomposition in part a). follows from $-K_S-vA\sim_{\mathbb{R}}(4-v)A+2B'+3B+2C+D$. A similar statement holds in other parts. We have $S_S(A)=\frac{4}{3}$. Thus, $\delta_P(S)\leq\frac{3}{4}$ for $P\in A$. Moreover for $P\in A\backslash B_1$ we have $h(v)\leq\frac{(4-v)(3v+4)}{32}$ if $v\in[0,4]$. So $S(W^A_{\bullet,\bullet};P)\leq 1\leq\frac{4}{3}$. We get that $\delta_P(S)=\frac{3}{4}$ for $P\in A\backslash B_1$. □

Lemma 6.1.20. *If P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

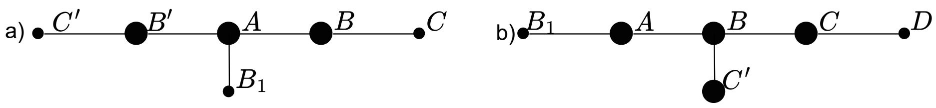


Figure 6.17: Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{2}{3}$

Then $\tau(A) = 3$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}(B+B') \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}(B+B') - (v-1)B_1 \text{ if } v \in [1,2], \\ -K_S - vA - (v-1)(B+B'+B_1) - (v-2)(C+C') \text{ if } v \in [2,3]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{2}(B+B') \text{ if } v \in [0,1], \\ \frac{v}{2}(B+B') + (v-1)B_1 \text{ if } v \in [1,2], \\ (v-1)(B+B'+B_1) + (v-2)(C+C') \text{ if } v \in [2,3]. \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}(2B+C+C') \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}(2B+C+C') - (v-1)B_1 \text{ if } v \in [1,2], \\ -K_S - vA - (v-1)(2B+C'+B_1) - (2v-3)C - (2v-4)D \text{ if } v \in [2,3]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{2}(2B+C+C') \text{ if } v \in [0,1], \\ \frac{v}{2}(2B+C+C') + (v-1)B_1 \text{ if } v \in [1,2], \\ (v-1)(2B+C'+B_1) + (2v-3)C + (2v-4)D \text{ if } v \in [2,3]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} (2-v)(2+v) \text{ if } v \in [0,1], \\ 5-2v \text{ if } v \in [1,2], \\ (3-v)^2 \text{ if } v \in [2,3]. \end{cases} \qquad P(v)\cdot A = \begin{cases} v \text{ if } v \in [0,1], \\ 1 \text{ if } v \in [1,2], \\ 3-v \text{ if } v \in [2,3]. \end{cases}$$

In this case $\delta_P(S) = \frac{2}{3}$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} (3-v)A + C' + 2B' + 2B + C' + 2B_1$. We have $S_S(A) = \frac{3}{2}$. Thus, $\delta_P(S) \leq \frac{2}{3}$ for $P \in A$. Note that fo $P \in A \backslash B$ we have:

$$h(v) \leq \begin{cases} \frac{v^2}{2} \text{ if } v \in [0,1], \\ v + \frac{1}{2} \text{ if } v \in [1,2], \\ \frac{(3-v)(v+1)}{2} \text{ if } v \in [2,3]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\leq 1\leq\frac{3}{2}$. Thus, $\delta_P(S)=\frac{2}{3}$ if $P\in A\backslash B$. □

Lemma 6.1.21. *If P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

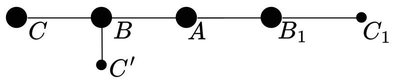


Figure 6.18: Dual graph: $(-K_S)^2=4$ and $\delta_P(S)=\frac{24}{37}$

Then $\tau(A)=3$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}B_1-\frac{v}{3}(2B+C) \text{ if } v\in\left[0,\frac{3}{2}\right],\\ -K_S-vA-\frac{v}{2}B_1-(v-1)(2B+C)-(2v-3)C' \text{ if } v\in\left[\frac{3}{2},2\right],\\ -K_S-vA-(v-1)(2B+C+B_1)-(2v-3)C'-(v-2)C_1 \text{ if } v\in[2,3].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{2}B_1+\frac{v}{3}(2B+C) \text{ if } v\in\left[0,\frac{3}{2}\right],\\ \frac{v}{2}B_1+(v-1)(2B+C)+(2v-3)C' \text{ if } v\in\left[\frac{3}{2},2\right],\\ (v-1)(2B+C+B_1)+(2v-3)C'+(v-2)C_1 \text{ if } v\in[2,3].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}4-\frac{5v^2}{6} \text{ if } v\in\left[0,\frac{3}{2}\right],\\ 7-4v+\frac{v^2}{2} \text{ if } v\in\left[\frac{3}{2},2\right],\\ (3-v)^2 \text{ if } v\in[2,3].\end{cases}\quad P(v)\cdot A\begin{cases}\frac{5v}{6} \text{ if } v\in\left[0,\frac{3}{2}\right],\\ 2-\frac{v}{2} \text{ if } v\in\left[\frac{3}{2},2\right],\\ 3-v \text{ if } v\in[2,3].\end{cases}$$

In this case $\delta_P(S)=\frac{24}{37}$ if $P\in A\backslash B$.

Proof. The Zariski Decomposition follows from $-K_S-vA\sim_{\mathbb{R}}(3-v)A+2B_1+C_1+4B+2C+3C'$. We have $S_S(A)=\frac{37}{24}$. Thus, $\delta_P(S)\leq\frac{24}{37}$ for $P\in A$. Note that for $P\in A\backslash B$ we have:

$$h(v)\leq\begin{cases}\frac{55v^2}{72} \text{ if } v\in\left[0,\frac{3}{2}\right],\\ \frac{(4-v)(4+v)}{8} \text{ if } v\in\left[\frac{3}{2},2\right],\\ \frac{(3-v)(v+1)}{2} \text{ if } v\in[2,3].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\leq\frac{13}{12}\leq\frac{37}{24}$. Thus, $\delta_P(S)=\frac{24}{37}$ if $P\in A\backslash B$. □

Lemma 6.1.22. *If P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

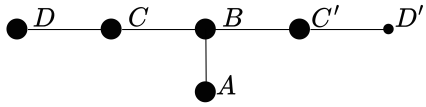


Figure 6.19: Dual graph: $(-K_S)^2=4$ and $\delta_P(S)=\frac{9}{14}$

Then $\tau(A)=3$ *and the Zariski Decomposition of the divisor* $-K_S-vA$ *is given by:*

$$P(v)=\begin{cases}-K_S-vA-\frac{v}{5}(6B+4C+3C'+2D) \text{ if } v\in\big[0,\frac{5}{3}\big],\\ -K_S-vA-(v-1)(3B+2C+D)-(3v-4)C'-(3v-5)D' \text{ if } v\in\big[\frac{5}{3},3\big].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{5}(6B+4C+3C'+2D) \text{ if } v\in\big[0,\frac{5}{3}\big],\\ (v-1)(3B+2C+D)+(3v-4)C'+(3v-5)D' \text{ if } v\in\big[\frac{5}{3},3\big].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}4-\frac{4v^2}{5} \text{ if } v\in\big[0,\frac{5}{3}\big],\\ (3-v)^2 \text{ if } v\in\big[\frac{5}{3},3\big].\end{cases}\qquad P(v)\cdot A=\begin{cases}\frac{4v}{5} \text{ if } v\in\big[0,\frac{5}{3}\big],\\ 3-v \text{ if } v\in\big[\frac{5}{3},3\big].\end{cases}$$

In this case $\delta_P(S)=\frac{9}{14}$ *if* $P\in A\backslash B$.

Proof. The Zariski Decomposition follows from $-K_S-vA\sim_{\mathbb{R}}(3-v)A+4D'+5C'+6B+4C+2D$. We have $S_S(A)=\frac{14}{9}$. Thus, $\delta_P(S)\leq\frac{9}{14}$ for $P\in A$. Note that if $P\in A\backslash B$ then

$$h(v)=\begin{cases}\frac{8v^2}{25} \text{ if } v\in\big[0,\frac{5}{3}\big],\\ \frac{(3-v)^2}{2} \text{ if } v\in\big[\frac{5}{3},3\big].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\leq\frac{14}{9}$. Thus, $\delta_P(S)=\frac{9}{14}$ if $P\in A\backslash B$. □

Lemma 6.1.23. *If* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

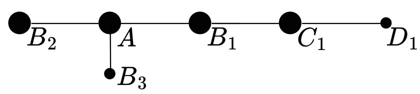


Figure 6.20: Dual graph: $(-K_S)^2=4$ and $\delta_P(S)=\frac{6}{11}$

Then $\tau(A)=4$ *and the Zariski Decomposition of the divisor* $-K_S-vA$ *is given by:*

$$P(v)=\begin{cases}-K_S-vA-\frac{v}{3}(2B_1+C_1)-\frac{v}{2}B_2 \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{3}(2B_1+C_1)-\frac{v}{2}B_2-(v-1)B_3 \text{ if } v\in[1,3],\\ -K_S-vA-(v-1)(B_1+B_3)-(v-2)C_1-(v-3)D_1-\frac{v}{2}B_2 \text{ if } v\in[3,4].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{3}(2B_1+C_1)+\frac{v}{2}B_2 \text{ if } v\in[0,1],\\ \frac{v}{3}(2B_1+C_1)+\frac{v}{2}B_2+(v-1)B_3 \text{ if } v\in[1,3],\\ (v-1)(B_1+B_3)+(v-2)C_1+(v-3)D_1+\frac{v}{2}B_2 \text{ if } v\in[3,4].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}4-\frac{5v^2}{6} \text{ if } v\in[0,1],\\ 5-2v+\frac{v^2}{6} \text{ if } v\in[1,3],\\ \frac{(4-v)^2}{2} \text{ if } v\in[3,4].\end{cases}\quad P(v)\cdot A=\begin{cases}\frac{5v}{6} \text{ if } v\in[0,1],\\ 1-\frac{v}{6} \text{ if } v\in[1,3],\\ 2-\frac{v}{2} \text{ if } v\in[3,4].\end{cases}$$

In this case $\delta_P(S)=\frac{6}{11}$ if $P\in A$.

Proof. The Zariski Decomposition follows from $-K_S-vA\sim_{\mathbb{R}}(4-v)A+3B_1+2C_1+D_1+2B_2+3B_3$. We have $S_S(A)=\frac{11}{6}$. Thus, $\delta_P(S)\le\frac{6}{11}$ for $P\in A$. Note that if $P\in A\backslash(B_1\cup B_3)$ or $P\in A\cap B_1$ or $P\in A\cap B_3$ we have:

$$h(v)\le\begin{cases}\frac{55v^2}{72} \text{ if } v\in[0,1],\\ \frac{(6-v)(5v+6)}{72} \text{ if } v\in[1,3],\\ \frac{(4-v)(v+4)}{8} \text{ if } v\in[3,4].\end{cases}\quad \text{or } h(v)\le\begin{cases}\frac{65v^2}{72} \text{ if } v\in[0,1],\\ \frac{(6-v)(7v+6)}{72} \text{ if } v\in[1,3],\\ \frac{3(4-v)v}{8} \text{ if } v\in[3,4].\end{cases}$$

$$\text{or } h(v)\le\begin{cases}\frac{25v^2}{72} \text{ if } v\in[0,1],\\ \frac{(6-v)(11v-6)}{72} \text{ if } v\in[1,3],\\ \frac{3(4-v)v}{8} \text{ if } v\in[3,4].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\le\frac{11}{9}\le\frac{11}{6}$ or $S(W^A_{\bullet,\bullet};P)\le\frac{37}{24}\le\frac{11}{6}$ $S(W^A_{\bullet,\bullet};P)\le\frac{29}{24}\le\frac{11}{6}$. Thus, $\delta_P(S)=\frac{6}{11}$ if $P\in A$. □

Lemma 6.1.24. *If P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

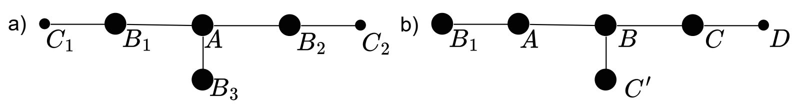


Figure 6.21: Dual graph: $(-K_S)^2=4$ and $\delta_P(S)=\frac{1}{2}$ with $\tau(A)=4$

Then $\tau(A)=4$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

a).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}(B_1+B_2+B_3) \text{ if } v\in[0,2],\\ -K_S-vA-\frac{v}{2}B_1-(v-1)(B_2+B_3)-(v-2)(C_2+C_3) \text{ if } v\in[2,4].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{2}(B_1+B_2+B_3) \text{ if } v\in[0,2],\\ \frac{v}{2}B_1+(v-1)(B_2+B_3)+(v-2)(C_2+C_3) \text{ if } v\in[2,4].\end{cases}$$

b). $$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}(B_1+2B+C+C') \text{ if } v\in[0,2],\\ -K_S-vA-\frac{v}{2}E_1-(v-1)(2B+C')-(2v-3)C-(2v-4)D \text{ if } v\in[2,4].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{2}(B_1+2B+C+C') \text{ if } v\in[0,2],\\ \frac{v}{2}B_1+(v-1)(2B+C')+(2v-3)C+(2v-4)D \text{ if } v\in[2,4].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}4-\frac{v^2}{2} \text{ if } v\in[0,2],\\ \frac{(4-v)^2}{2} \text{ if } v\in[2,4].\end{cases}\quad P(v)\cdot A=\begin{cases}\frac{v}{2} \text{ if } v\in[0,2]\\ 2-\frac{v}{2} \text{ if } v\in[2,4].\end{cases}$$

In this case $\delta_P(S)=\frac{1}{2}$ *if* $P\in A\backslash B$.

Proof. The Zariski Decomposition in part a). follows from $-K_S-vA\sim_{\mathbb{R}}(4-v)A+3B_1+2C_1+3B_2+2C_2+2B_3$. A similar statement holds in other parts. Thus, $S_S(A)=2$ Thus, $\delta_P(S)\le\frac{1}{2}$ for $P\in A$. Note that we have that

$$h(v)\le\begin{cases}\frac{3v^2}{8} \text{ if } v\in[0,2],\\ \frac{(4-v)(v+4)}{8} \text{ if } v\in[2,4].\end{cases}\quad \text{or } h(v)=\begin{cases}\frac{3v^2}{8} \text{ if } v\in[0,2],\\ \frac{3(4-v)v}{8} \text{ if } v\in[2,4].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\le\frac{3}{2}\le 2$ or $S(W^A_{\bullet,\bullet};P)\le\frac{4}{3}\le 2$. Thus, $\delta_P(S)=\frac{1}{2}$. □

Lemma 6.1.25. *If* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

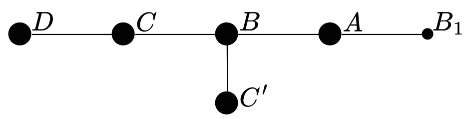


Figure 6.22: Dual graph: $(-K_S)^2=4$ and $\delta_P(S)=\frac{1}{2}$ with $\tau(A)=5$

Then $\tau(A)=5$ *and the Zariski Decomposition of the divisor* $-K_S-vA$ *is given by:*

$$P(v)=\begin{cases}-K_S-vA-\frac{v}{5}(6B+4C+3C'+2D) \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{5}(6B+4C+3C'+2D)-(v-1)B_1 \text{ if } v\in[1,5].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{5}(6B+4C+3C'+2D) \text{ if } v\in[0,1],\\ \frac{v}{5}(6B+4C+3C'+2D)+(v-1)B_1 \text{ if } v\in[1,5].\end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 4 - \frac{4v^2}{5} \text{ if } v \in [0,1], \\ \frac{(5-v)^2}{5} \text{ if } v \in [1,5]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{4v}{5} \text{ if } v \in [0,1], \\ 1 - \frac{v}{5} \text{ if } v \in [1,5]. \end{cases}$$

In this case $\delta_P(S) = \frac{1}{2}$ *if* $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (5-v)A + 3C' + 6B + 4C + 2D + 4B_1$. We have $S_S(A) = 2$. Thus, $\delta_P(S) \le \frac{1}{2}$ for $P \in A$. Note that for $P \in A \backslash B$ we have:

$$h(v) = \begin{cases} \frac{8v^2}{25} \text{ if } v \in [0,1], \\ \frac{(5-v)(9v-5)}{50} \text{ if } v \in [1,5]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \le \frac{4}{3} \le 2$. Thus, $\delta_P(S) = \frac{1}{2}$ if $P \in A \backslash B$. □

Lemma 6.1.26. *If* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

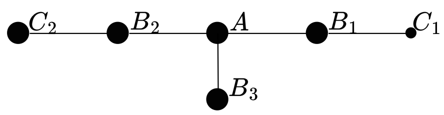


Figure 6.23: Dual graph: $(-K_S)^2 = 4$ and $\delta_P(S) = \frac{3}{8}$

Then $\tau(A) = 6$ *and the Zariski Decomposition of the divisor* $-K_S - vA$ *is given by:*

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{6}(4C_2 + 2B_2 + 3B_3 + 3B_1) \text{ if } v \in [0,2], \\ -K_S - vA - \frac{v}{6}(4C_2 + 2B_2 + 3B_3) - (v-1)B_1 - (v-2)C_1 \text{ if } v \in [2,6]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(4C_2 + 2B_2 + 3B_3 + 3B_1) \text{ if } v \in [0,2], \\ \frac{v}{6}(4C_2 + 2B_2 + 3B_3) + (v-1)B_1 + (v-2)C_1 \text{ if } v \in [2,6]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 4 - \frac{v^2}{3} \text{ if } v \in [0,2], \\ \frac{(6-v)^2}{6} \text{ if } v \in [2,6]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{v}{3} \text{ if } v \in [0,2], \\ 1 - \frac{v}{6} \text{ if } v \in [2,6]. \end{cases}$$

In this case $\delta_P(S) = \frac{3}{8}$ *if* $P \in A$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (6-v)A + 5B_1 + 4C_1 + 4B_2 + 2C_2 + 3B_3$. We have $S_S(A) = \frac{8}{3}$. Thus, $\delta_P(S) \leq \frac{3}{8}$ for $P \in A$. Note that

$$h(v) \leq \begin{cases} \frac{5v^2}{18} \text{ if } v \in [0,2], \\ \frac{(6-v)(7v+6)}{72} \text{ if } v \in [2,6]. \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{2v^2}{9} \text{ if } v \in [0,2], \\ \frac{(6-v)(5v+6)}{72} \text{ if } v \in [2,6]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{8}{3}$ or $S(W^A_{\bullet,\bullet};P) \leq \frac{14}{9} \leq \frac{8}{3}$. Thus, $\delta_P(S) = \frac{3}{8}$ if $P \in A$. □

6.2 Finding δ-invariants for degree 4

Let X be a singular del Pezzo surface of degree 4 with and S be a minimal resolution of X. Then there are several possible cases:

I. X has an $\mathbb{A}_1$ singularity and contains 12 lines. In this case, we let E be the exceptional divisor, L_i, $L_{i,j}$ for $i \in \{1,2,3,4\}$, $j \in \{1,2\}$ be the lines on S,

II. X has two $\mathbb{A}_1$ singularities and contains 9 lines. In this case, we let E_i be for $i \in \{1,2\}$ be the exceptional divisors, L_{12}, $L_{i,j}$, $L_{12,ij}$ for $i,j \in \{1,2\}$ be the lines on S,

III. X has two $\mathbb{A}_1$ singularities and contains 8 lines. In this case, we let E_i be for $i \in \{1,2\}$ be the exceptional divisors, $L_{i,j}$, for $i \in \{1,2\}$, $j \in \{1,2,3,4\}$ be the lines on S,

IV. X has three $\mathbb{A}_1$ singularities and contains 6 lines. In this case, we let E_1, E_1' and E_2 be the exceptional divisors, L_{12}, L_{12}', $L_{1,i}$ and $L_{1,i}'$ for $i \in \{1,2\}$ be the lines on S,

V. X has four $\mathbb{A}_1$ singularities and contains 4 lines. In this case, we let E_i for $i \in \{1,2,3,4\}$ be the exceptional divisors, L_{12}, L_{23}, L_{34}, L_{14} be the lines on S,

VI. X has an $\mathbb{A}_2$ singularity and contains 8 lines. In this case, we let E and E' be the exceptional divisors, L_i, $L_{i,1}$ and $L_{i,1}'$ for $i \in \{1,2\}$ be the lines on S,

VII. X has $\mathbb{A}_2$ and $\mathbb{A}_1$ singularities and contains 6 lines. In this case, we let E_1, E_1' and E_2 be the exceptional divisors, $L_{1,i}$, $L_{12,i}$ for $i \in \{1,2\}$, L_2 and L_{12} be the lines on S,

VIII. X has $\mathbb{A}_2$ and two $\mathbb{A}_1$ singularities and contains 4 lines. In this case, we let E_i and E_i' for $i \in \{1,2\}$ be the exceptional divisors, L_2, L_2', L_{12} and L_{12}' be the lines on S,

IX. X has an $\mathbb{A}_3$ singularity and contains 5 lines. In this case, we let E_1, E_1' and E_2 be the exceptional divisors, L_2, L_1, L_1', $L_{1,1}$ and $L_{1,1}'$ be the lines on S,

X. X has an $\mathbb{A}_3$ singularity and contains 4 lines. In this case, we let E_1, E_1' and E_2 be the exceptional divisors, $L_{1,i}$ and $L_{1,i}'$ for $i \in \{1,2\}$ be the lines on S,

XI. X has $\mathbb{A}_3$ and $\mathbb{A}_1$ singularities and contains 3 lines. In this case, we let E_1, E_1', E_2 and E_3 be the exceptional divisors, L_{13}, $L_{1,1}$ and $L_{1,2}$ be the lines on S,

XII. X has $\mathbb{A}_3$ and two $\mathbb{A}_1$ singularities and contains 2 lines. In this case, we let E_1, E_1', E_2, E_3 and E_3' be the exceptional divisors, L_{13} and L_{13}' be the lines on S,

XIII. X has an $\mathbb{A}_4$ singularity and contains 3 lines. In this case, we let E_i for $i \in \{1,2,3,4\}$ be the exceptional divisors, L_2, L_4 and $L_{4,1}$ be the lines on S,

XIV. X has an $\mathbb{D}_4$ singularity and contains 2 lines. In this case, we let E, E_1, E_1' and E_2 be the exceptional divisors, L_1 and L_1' be the lines on S,

XV. X has an $\mathbb{D}_5$ singularity and contains 1 line. In this case, we let E and E_i for $i \in \{1,2,3,4\}$ be the exceptional divisors, L_4 be a line on S.

such that the dual graph of the (-1)-curves and (-2)-curves on S is given on the picture below:

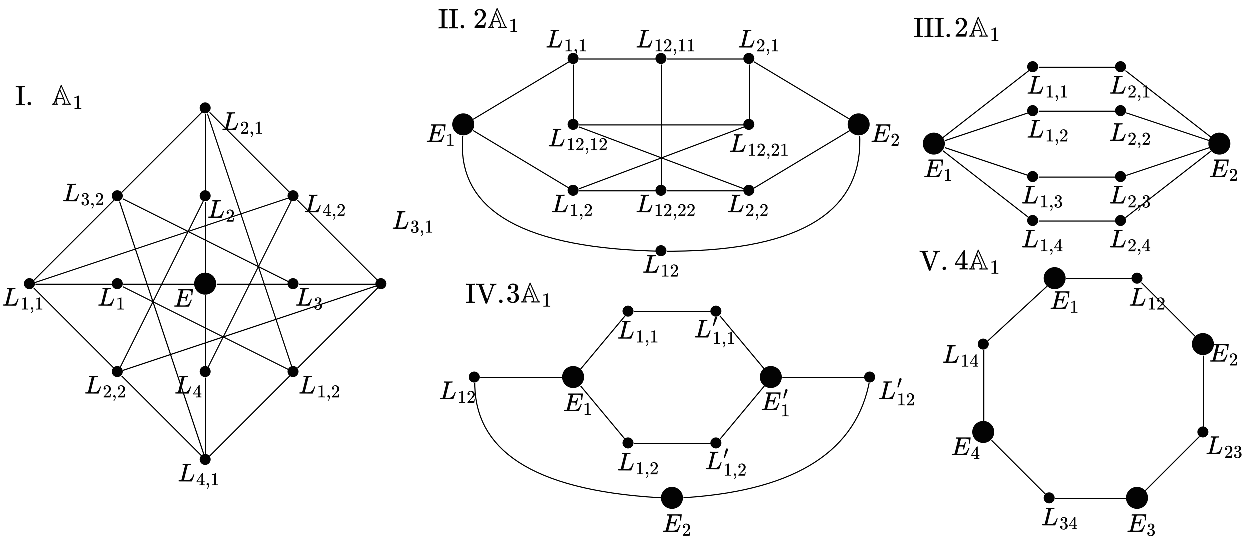


Figure 6.24: Du Val del Pezzo surfaces with $(-K_S)^2 = 4$ (pic. 1/3)

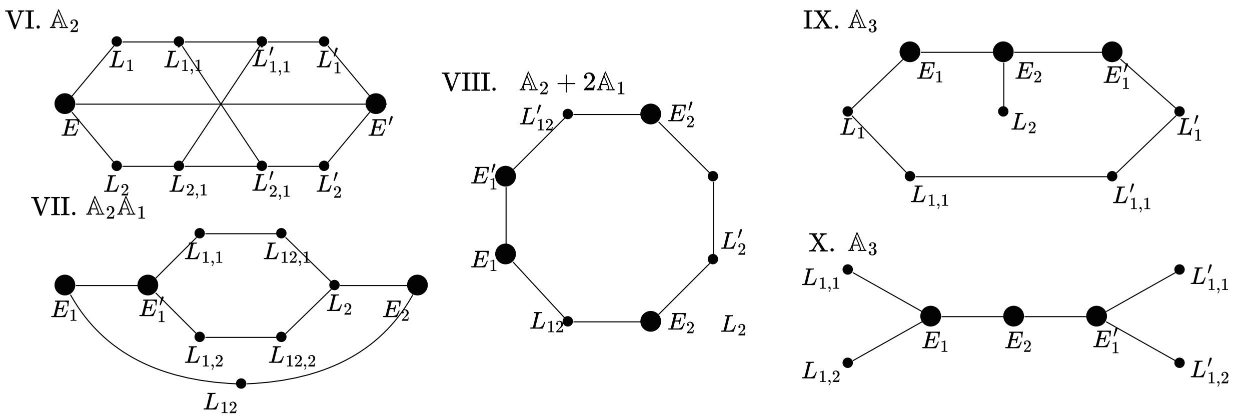


Figure 6.25: Du Val del Pezzo surfaces with $(-K_S)^2 = 4$ (pic. 2/3)

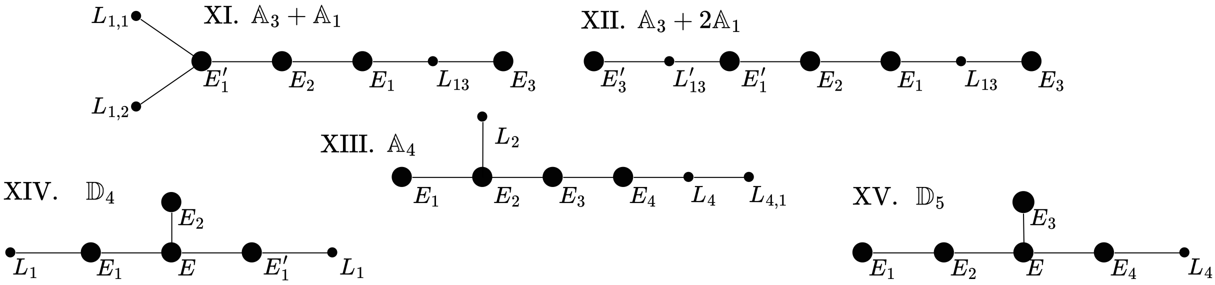


Figure 6.26: Du Val del Pezzo surfaces with $(-K_S)^2 = 4$ (pic. 2/2)

One has:

I. $\delta(X) = 1$ since depending on the position of point $P \in S$ we have

P	E	$\left(\bigcup_{i\in\{1,2,3,4\}} L_i\right)\backslash E$	2 curves in $\mathbf{L}_1$	1 curve in $\mathbf{L}_1\backslash\left(\bigcup_{i\in\{1,2,3,4\}} L_i\right)$	o/w
$\delta_P(S)$	1	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{18}{13}$	$\frac{3}{2}$

Table 6.6: Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{A}_1$ singularity

where $\mathbf{L}_1 := \{L_{i,j} | i \in \{1,2,3,4\}, j \in \{1,2\}\}$.

II. $\delta(X) = 1$ since depending on the position of point $P \in S$ we have

P	$\mathbf{E}_2 \cup L_{12}$	$\left(\bigcup_{i,j\in\{1,2\}} L_{i,j}\right)\backslash \mathbf{E}_2$	$\mathbf{L}_2$	$\mathbf{L}_{12}\backslash\left(\mathbf{L}_2 \cup \bigcup_{i,j\in\{1,2\}} L_{i,j}\right)$	o/w
$\delta_P(S)$	1	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{18}{13}$	$\frac{3}{2}$

Table 6.7: Local δ-invariants: $(-K_S)^2 = 4$ and $2\mathbb{A}_1$ singularities (9 lines)

where $\mathbf{E}_2 := E_1 \cup E_2$, $\mathbf{L}_2 := (L_{12,11} \cap L_{12,22}) \cup (L_{12,12} \cap L_{12,21})$, $\mathbf{L}_{12} := \bigcup_{i,j\in\{1,2\}} L_{12,ij}$.

III. $\delta(X) = 1$ since depending on the position of point $P \in S$ we have

P	$E_1 \cup E_2$	$\left(\bigcup_{i\in\{1,2\},j\in\{1,2,3,4\}} L_{i,j}\right)\backslash(E_1 \cup E_2)$	o/w
$\delta_P(S)$	1	$\frac{6}{5}$	$\frac{3}{2}$

Table 6.8: Local δ-invariants: $(-K_S)^2 = 4$ and $2\mathbb{A}_1$ singularities (8 lines)

IV. $\delta(X) = 1$ since depending on the position of point $P \in S$ we have

P	$E_1 \cup E_1' \cup E_2 \cup L_{12} \cup L_{12}'$	$\left(\bigcup_{i\in\{1,2\}} L_{1,i} \cup L_{1,i}'\right)\backslash(E_1 \cup E_1')$	o/w
$\delta_P(S)$	1	$\frac{6}{5}$	$\frac{3}{2}$

Table 6.9: Local δ-invariants: $(-K_S)^2 = 4$ and $3\mathbb{A}_1$ singularities

V. $\delta(X) = 1$ since depending on the position of point $P \in S$ we have

P	$\left(\bigcup_{i\in\{1,2,3,4\}} E_i\right) \cup L_{12} \cup L_{23} \cup L_{34} \cup L_{14}$	o/w
$\delta_P(S)$	1	$\frac{3}{2}$

Table 6.10: Local δ-invariants: $(-K_S)^2 = 4$ and $4\mathbb{A}_1$ singularities

VI. $\delta(X) = \frac{6}{7}$ since depending on the position of point $P \in S$ we have

P	$\mathbf{E}_6$	$\left(\bigcup_{i\in\{1,2\}} L_i \cup L_i'\right) \setminus \mathbf{E}_6$	$\mathbf{L}_6$	$\left(\bigcup_{i\in\{1,2\}} L_{i,1} \cup L_{i,1}'\right) \setminus \left(\mathbf{L}_6 \cup \mathbf{E}_6\right)$	o/w
$\delta_P(S)$	$\frac{6}{7}$	$\frac{8}{7}$	$\frac{4}{3}$	$\frac{18}{13}$	$\frac{3}{2}$

where $\mathbf{E}_6 := E \cup E'$, $\mathbf{L}_6 := \left(L_{1,1} \cap L_{1,1}'\right) \cup \left(L_{1,1} \cap L_{2,1}'\right) \cup \left(L_{2,1} \cap L_{1,1}'\right) \cup \left(L_{2,1} \cap L_{2,1}'\right)$.

Table 6.11: Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{A}_2$ singularity

VII. $\delta(X) = \frac{6}{7}$ since depending on the position of point $P \in S$ we have

P	$E_1 \cup E_1' \cup L_{12}$	$E_2 \setminus L_{12}$	$\mathbf{L}_7 \setminus E_1'$	$L_2 \setminus E_2$	$(L_{12,1} \cup L_{12,2}) \setminus (\mathbf{L}_7 \cup L_2)$	o/w
$\delta_P(S)$	$\frac{6}{7}$	1	$\frac{8}{7}$	$\frac{6}{5}$	$\frac{18}{13}$	$\frac{3}{2}$

where $\mathbf{L}_7 := L_{1,1} \cup L_{1,2}$.

Table 6.12: Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{A}_2\mathbb{A}_1$ singularities

VIII. $\delta(X) = \frac{6}{7}$ since depending on the position of point $P \in S$ we have

P	$E_1 \cup E_1' \cup L_{12} \cup L_{12}'$	$(E_2 \cup E_2') \setminus (L_{12} \cup L_{12}')$	$(L_2 \cup L_2') \setminus (E_2 \cup E_2')$	o/w
$\delta_P(S)$	$\frac{6}{7}$	1	$\frac{6}{5}$	$\frac{3}{2}$

Table 6.13: Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{A}_2 2\mathbb{A}_1$ singularities

IX. $\delta(X) = \frac{3}{4}$ since depending on the position of point $P \in S$ we have

P	E_2	$\mathbf{E}_9 \setminus E_2$	$(L_1 \cup L_1') \setminus \mathbf{E}_9$	$L_2 \setminus E_2$	$\mathbf{L}_9$	$(L_{1,1} \cup L_{1,1}') \setminus \mathbf{L}_9$	o/w
$\delta_P(S)$	$\frac{2}{3}$	$\frac{24}{29}$	$\frac{12}{11}$	1	$\frac{4}{3}$	$\frac{18}{13}$	$\frac{3}{2}$

where $\mathbf{E}_9 := E_1 \cup E_1'$, $\mathbf{L}_9 := L_{1,1} \cap L_{1,1}'$.

Table 6.14: Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{A}_3$ singularity (5 lines)

X. $\delta(X) = \frac{3}{4}$ since depending on the position of point $P \in S$ we have

P	$E_1 \cup E_1' \cup E_2$	$\left(\bigcup_{i\in\{1,2\}} L_{1,i} \cup L_{1,i}'\right) \setminus (E_1 \cup E_1')$	o/w
$\delta_P(S)$	$\frac{3}{4}$	$\frac{9}{8}$	$\frac{3}{2}$

Table 6.15: Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{A}_3$ singularity (4 lines)

XI. $\delta(X) = \frac{3}{4}$ since depending on the position of point $P \in S$ we have

P	$E_1 \cup E_1' \cup E_2 \cup L_{13}$	$E_3 \backslash L_{13}$	$(L_{1,1} \cup L_{1,2}) \backslash E_1'$	o/w
$\delta_P(S)$	$\frac{3}{4}$	1	$\frac{9}{8}$	$\frac{3}{2}$

Table 6.16: Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{A}_3\mathbb{A}_1$ singularities

XII. $\delta(X) = \frac{3}{4}$ since

P	$E_1 \cup E_1' \cup E_2 \cup L_{13} \cup L_{13}'$	$(E_3 \cup E_3') \backslash (L_{13} \cup L_{13}')$	o/w
$\delta_P(S)$	$\frac{3}{4}$	1	$\frac{3}{2}$

Table 6.17: Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{A}_3 2\mathbb{A}_1$ singularities

XIII. $\delta(X) = \frac{6}{11}$ since depending on the position of point $P \in S$ we have

P	E_2	$E_3 \backslash E_2$	$E_4 \backslash E_3$	$L_2 \backslash E_2$	$E_1 \backslash E_2$	$L_4 \backslash E_4$	$L_{4,1} \backslash L_4$	o/w
$\delta_P(S)$	$\frac{6}{11}$	$\frac{24}{37}$	$\frac{4}{5}$	$\frac{24}{29}$	$\frac{9}{11}$	$\frac{24}{23}$	$\frac{18}{13}$	$\frac{3}{2}$

Table 6.18: Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{A}_4$ singularity

XIV. $\delta(X) = \frac{1}{2}$ since depending on the position of point $P \in S$ we have

P	E	$(E_1 \cup E_1') \backslash E$	$E_2 \backslash E$	$(L_1 \cup L_1') \backslash (E_1 \cup E_1')$	o/w
$\delta_P(S)$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	1	$\frac{3}{2}$

Table 6.19: Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{D}_4$ singularity

XV. $\delta(X) = \frac{3}{8}$ since depending on the position of point $P \in S$ we have

P	E_3	$(E_2 \cup E_4) \backslash E_3$	$E \backslash E_3$	$(E_1 \cup L_4) \backslash (E_2 \cup E_4)$	o/w
$\delta_P(S)$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{9}{14}$	$\frac{3}{4}$	$\frac{3}{2}$

Table 6.20: Local δ-invariants: $(-K_S)^2 = 4$ and $\mathbb{D}_5$ singularity

Proof. We prove each case separately using lemmas from the previous section.

I. If $P \in E$, the assertion follows from Lemma 6.1.9 [a).]. If $P \in \left(\bigcup_{i\in\{1,2,3,4\}} L_i\right) \backslash E$, the assertion follows from Lemma 6.1.4 [a).]. If P is the intersection of two (-1)-curves in $\left\{L_{i,j} \middle| i \in \{1,2,3,4\}, j \in \{1,2\}\right\}$, the assertion follows from Lemma 6.1.3. If P belongs to exactly one curve in $\left\{L_{i,j} \middle| i \in \{1,2,3,4\}, j \in \{1,2\}\right\} \backslash \left(\bigcup_{i\in\{1,2,3,4\}} L_i\right)$, the assertion follows from Lemma 6.1.2. If P is a general point, the assertion follows from Lemma 6.1.1.

II. If $P \in E_1 \cup E_2$, the assertion follows from Lemma 6.1.9 [b).]. If $P \in L_{12} \backslash (E_1 \cup E_2)$, the assertion follows from Lemma 6.1.10. If $P \in \left(\bigcup_{i,j\in\{1,2\}} L_{i,j}\right) \backslash (E_1 \cup E_2)$, the assertion follows from Lemma 6.1.4 [a).]. If $P \in \left(L_{12,11} \cap L_{12,22}\right) \cup \left(L_{12,12} \cap L_{12,21}\right)$, the assertion follows from Lemma 6.1.3. If $P \in \left(\bigcup_{i,j\in\{1,2\}} L_{12,ij}\right) \backslash \Big(\left(L_{12,11} \cap L_{12,22}\right) \cup \left(L_{12,12} \cap L_{12,21}\right) \cup \bigcup_{i,j\in\{1,2\}} L_{i,j}\Big)$, the assertion follows from Lemma 6.1.2. If P is a general point, the assertion follows from Lemma 6.1.1.

III. If $P \in E_1 \cup E_2$, the assertion follows from Lemma 6.1.9 [a).]. If $P \in \left(\bigcup_{i\in\{1,2\},j\in\{1,2,3,4\}} L_{i,j}\right) \backslash (E_1 \cup E_2)$, the assertion follows from Lemma 6.1.4 [b).]. If P is a general point, the assertion follows from Lemma 6.1.1.

IV. If $P \in E_2$, the assertion follows from Lemma 6.1.9 [c).]. If $P \in E_1 \cup E_1'$, the assertion follows from Lemma 6.1.9 [b).]. If $P \in (L_{12} \cup L_{12}') \backslash (E_1 \cup E_1' \cup E_2)$, the assertion follows from Lemma 6.1.10. If $P \in \left(\bigcup_{i\in\{1,2\}} L_{1,i} \cup L_{1,i}'\right) \backslash (E_1 \cup E_1')$, the assertion follows from Lemma 6.1.4 [b).]. If P is a general point, the assertion follows from Lemma 6.1.1.

V. If $P \in \bigcup_{i\in\{1,2,3,4\}} E_i$, the assertion follows from Lemma 6.1.9 [c).]. If $P \in (L_{12} \cup L_{23} \cup L_{34} \cup L_{14}) \backslash \left(\bigcup_{i\in\{1,2,3,4\}} E_i\right)$, the assertion follows from Lemma 6.1.10 [a).]. If P is a general point, the assertion follows from Lemma 6.1.1.

VI. If $P \in E \cup E'$, the assertion follows from Lemma 6.1.11 [a).]. If $P \in \left(\bigcup_{i\in\{1,2\}} L_i \cup L_i'\right) \backslash (E \cup E')$, the assertion follows from Lemma 6.1.5. If $P \in \left(L_{1,1} \cap L_{1,1}'\right) \cup \left(L_{1,1} \cap L_{2,1}'\right) \cup \left(L_{2,1} \cap L_{1,1}'\right) \cup \left(L_{2,1} \cap L_{2,1}'\right)$, the assertion follows from Lemma 6.1.3. If $P \in \left(\bigcup_{i\in\{1,2\}} L_{i,1} \cup L_{i,1}'\right) \backslash \Big(\left(L_{1,1} \cap L_{1,1}'\right) \cup \left(L_{1,1} \cap L_{2,1}'\right) \cup \left(L_{2,1} \cap L_{1,1}'\right) \cup \left(L_{2,1} \cap L_{2,1}'\right) \cup E \cup E'\Big)$, the assertion follows from Lemma 6.1.2. If P is a general point, the assertion follows from Lemma 6.1.1.

VII. If $P \in E_1'$, the assertion follows from Lemma 6.1.11 [a).]. If $P \in E_1$, the assertion follows from Lemma 6.1.11 [b).]. If $P \in L_{12} \backslash E_1$, the assertion follows from Lemma 6.1.12. If $P \in E_2 \backslash L_{12}$, the assertion follows from Lemma 6.1.9 [d).]. If $P \in (L_{1,1} \cup L_{1,2}) \backslash E_1$, the assertion follows from Lemma 6.1.5. If $P \in L_2 \backslash E_2$, the assertion follows from Lemma 6.1.4 [a).]. If $P \in (L_{12,1} \cup L_{12,2}) \backslash (L_{1,1} \cup L_{1,2} \cup L_2)$, the assertion follows from Lemma 6.1.2. If P is a general point, the assertion follows from Lemma 6.1.1.

VIII. If $P \in (L_{12} \cup L_{12}') \backslash (E_1 \cup E_1')$, the assertion follows from Lemma 6.1.12. If $P \in E_1 \cup E_1'$, the assertion follows from Lemma 6.1.11 [b).]. If $P \in (E_2 \cup E_2') \backslash (L_{12} \cup L_{12}')$, the assertion follows from Lemma 6.1.9 [d).]. If $P \in (L_2 \cup L_2') \backslash (E_2 \cup E_2')$, the assertion follows from Lemma 6.1.4 [b).]. If P is a general point, the assertion follows from Lemma 6.1.1.

IX. If $P \in E_2$, the assertion follows from Lemma 6.1.17. If $P \in (E_1 \cup E_1') \backslash E_2$, the assertion follows from Lemma 6.1.13. If $P \in (L_1 \cup L_1') \backslash (E_1 \cup E_1')$, the assertion follows from Lemma 6.1.7. If $P \in L_2 \backslash E_2$, the assertion follows from Lemma 6.1.10 [b).]. If $P = L_{1,1} \cap L_{1,1}'$, the assertion follows from Lemma 6.1.3. If $P \in (L_{1,1} \cup L_{1,1}') \backslash (L_{1,1} \cap L_{1,1}')$, the assertion follows from Lemma 6.1.2. If P is a general point, the assertion follows from Lemma 6.1.1.

X. If $P \in E_2$, the assertion follows from Lemma 6.1.17. If $P \in (E_1 \cup E_1')\backslash E_2$, the assertion follows from Lemma 6.1.18 [a).]. If $P \in \left(\bigcup_{i\in\{1,2\}} L_{1,i} \cup L_{1,i}'\right)\backslash(E_1 \cup E_1')$, the assertion follows from Lemma 6.1.6. If P is a general point, the assertion follows from Lemma 6.1.1.

XI. If $P \in E_2$, the assertion follows from Lemma 6.1.20 [a).]. If $P \in E_1'\backslash E_2$, the assertion follows from Lemma 6.1.18 [a).]. If $P \in E_1\backslash E_2$, the assertion follows from Lemma 6.1.18 [b).]. If $P \in L_{13}\backslash E_1$, the assertion follows from Lemma 6.1.19 [a).]. If $P \in E_3\backslash L_{13}$, the assertion follows from Lemma 6.1.9 [e).]. If $P \in (L_{1,1} \cup L_{1,2})\backslash E_1'$, the assertion follows from Lemma 6.1.6. If P is a general point, the assertion follows from Lemma 6.1.1.

XII. If $P \in E_2$, the assertion follows from Lemma 6.1.17. If $P \in (E_1 \cup E_1')\backslash E_2$, the assertion follows from Lemma 6.1.18. If $P \in (L_{13} \cup L_{13}')\backslash(E_1 \cup E_1')$, the assertion follows from Lemma 6.1.19 [a).]. If $P \in (E_3 \cup E_3')\backslash(L_{13} \cup L_{13}')$, the assertion follows from Lemma 6.1.9 [e).]. If P is a general point, the assertion follows from Lemma 6.1.1.

XIII. If $P \in E_2$, the assertion follows from Lemma 6.1.23. If $P \in E_3\backslash E_2$, the assertion follows from Lemma 6.1.21. If $P \in E_1\backslash E_2$, the assertion follows from Lemma 6.1.15. If $P \in E_4\backslash E_3$, the assertion follows from Lemma 6.1.16. If $P \in L_2\backslash E_2$, the assertion follows from Lemma 6.1.14. If $P \in L_4\backslash E_4$, the assertion follows from Lemma 6.1.8. If $P \in L_{4,1}\backslash L_4$, the assertion follows from Lemma 6.1.2. If P is a general point, the assertion follows from Lemma 6.1.1.

XIV. If $P \in E$, the assertion follows from Lemma 6.1.24 [a).]. If $P \in (E_1 \cup E_1')\backslash E$, the assertion follows from Lemma 6.1.20 [b).]. If $P \in E_2\backslash E$, the assertion follows from Lemma 6.1.17 [b).]. If $P \in (L_1 \cup L_1')\backslash(E_1 \cup E_1')$, the assertion follows from Lemma 6.1.10 [c).]. If P is a general point, the assertion follows from Lemma 6.1.1.

XV. If $P \in E_3$, the assertion follows from Lemma 6.1.26. If $P \in E_2\backslash E_3$, the assertion follows from Lemma 6.1.24 [b).]. If $P \in E_4\backslash E_3$, the assertion follows from Lemma 6.1.25. If $P \in E_2\backslash E_3$, the assertion follows from Lemma 6.1.17 [c).]. If $P \in L_4\backslash E_4$, the assertion follows from Lemma 6.1.19 [b).]. If $P \in E\backslash E_3$, the assertion follows from Lemma 6.1.22. If P is a general point, the assertion follows from Lemma 6.1.1.

□

Chapter 7

Du Val del Pezzo Surfaces of Degree 3

In (Araujo et al., 2023, Lemma 2.13) it was proven that $\delta(X) = \frac{3}{2}$ when X is a smooth del Pezzo surface of degree 3 which contains an Eckardt point and $\delta(X) = \frac{27}{17}$ when X is a smooth del Pezzo surface of degree 3 which does not contain an Eckardt point. In this section, we compute δ-invariants of Du Val del Pezzo surfaces of degree 3.

MAIN THEOREM Let X be a singular Du Val del Pezzo surface of degree 3. Then the δ-invariant of X is uniquely determined by the degree of X, the number of lines on X, and the type of singularities on X which is given in the following table:

K_X^2	# lines	Sing(X)	$\delta(X)$
3	21	$\mathbb{A}_1$	$\frac{6}{5}$
3	16	$2\mathbb{A}_1$	$\frac{6}{5}$
3	12	$3\mathbb{A}_1$	$\frac{6}{5}$
3	9	$4\mathbb{A}_1$	$\frac{6}{5}$
3	15	$\mathbb{A}_2$	1
3	11	$\mathbb{A}_2+\mathbb{A}_1$	1
3	8	$\mathbb{A}_2+2\mathbb{A}_1$	1
3	7	$2\mathbb{A}_2$	1
3	5	$2\mathbb{A}_2+\mathbb{A}_1$	1
3	3	$3\mathbb{A}_2$	1
3	10	$\mathbb{A}_3$	$\frac{9}{11}$
3	7	$\mathbb{A}_3+\mathbb{A}_1$	$\frac{9}{11}$
3	5	$\mathbb{A}_3+2\mathbb{A}_1$	$\frac{9}{11}$
3	6	$\mathbb{A}_4$	$\frac{9}{13}$
3	4	$\mathbb{A}_4+\mathbb{A}_1$	$\frac{9}{13}$
3	3	$\mathbb{A}_5$	$\frac{3}{5}$
3	2	$\mathbb{A}_5+\mathbb{A}_1$	$\frac{3}{5}$
3	6	$\mathbb{D}_4$	$\frac{3}{5}$
3	3	$\mathbb{D}_5$	$\frac{9}{19}$
3	1	$\mathbb{E}_6$	$\frac{1}{3}$

Table 7.2: δ-invariants of Du Val del Pezzo surfaces of degree 3

7.1 General results for degree 3

Let X be a del Pezzo surface of degree 3 with at most Du Val singularities. Let S be a weak resolution of X. We will call an image of a (-1)-curve in S on X **a line** as was done in Cheltsov and Prokhorov (2021).

Lemma 7.1.1. *Assume that the point Q is not contained in any line that passes through a singular point of X. Then $\delta_Q(X) \geq \frac{3}{2}$.*

Proof. Follows from the proof of Lemma (Araujo et al., 2023, 2.13). □

Now we consider a curve A on S. Small circles correspond to a (-1)-curves and large circles correspond to a (-2)-curves on dual graphs.

Lemma 7.1.2. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph: Then $\tau(A)=\frac{4}{3}$ and the Zariski Decomposition of*

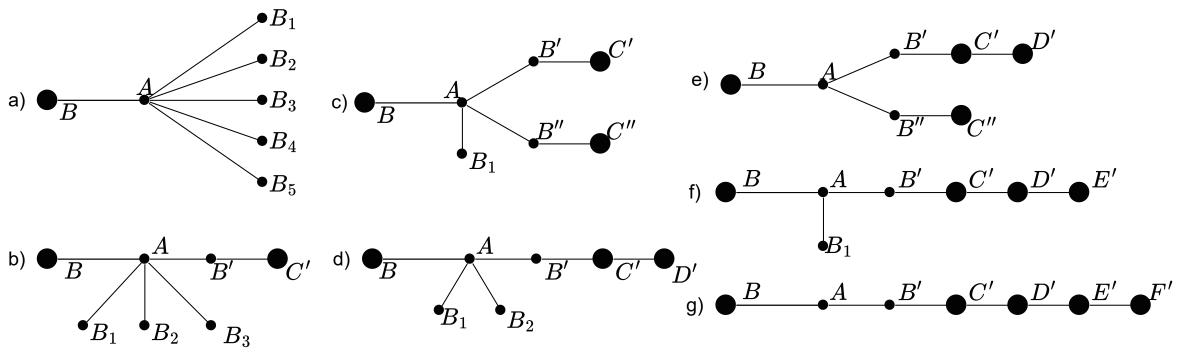


Figure 7.1: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{27}{17}$

the divisor $-K_S - vA$ is given by:

a).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B - (v-1)(B_1+B_2+B_3+B_4+B_5) \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}B \text{ if } v \in [0,1], \\ \frac{v}{2}B + (v-1)(B_1+B_2+B_3+B_4+B_5) \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

b).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B - (v-1)(2B'+C'+B_1+B_2+B_3) \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}B \text{ if } v \in [0,1], \\ \frac{v}{2}B + (v-1)(2B'+C'+B_1+B_2+B_3) \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

c).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B - (v-1)(B_1+2B'+C'+2B''+C'') \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}B \text{ if } v \in [0,1], \\ \frac{v}{2}B + (v-1)(B_1+2B'+C'+2B''+C'') \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

d).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B - (v-1)(B_1+B_2+3B'+2C'+D') \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}B \text{ if } v \in [0,1], \\ \frac{v}{2}B + (v-1)(B_1+B_2+3B'+2C'+D') \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

e).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B - (v-1)(3B'+2C'+D'+2B''+C'') \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ \frac{v}{2}B + (v-1)(3B' + 2C' + D' + 2B'' + C'') \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

f). $$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B - (v-1)(B_1 + 4B' + 3C' + 2D' + E') \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}B \text{ if } v \in [0,1], \\ \frac{v}{2}B + (v-1)(B_1 + 4B' + 3C' + 2D' + E') \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

g). $$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B - (v-1)(5B' + 4C' + 3D' + 2E' + F') \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}B \text{ if } v \in [0,1], \\ \frac{v}{2}B + (v-1)(5B' + 4C' + 3D' + 2E' + F') \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

Moreover:

$$(P(v))^2 = \begin{cases} 3 - 2v - \frac{v^2}{2} \text{ if } v \in [0,1], \\ \frac{(4-3v)^2}{2} \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} 1 + \frac{v}{2} \text{ if } v \in [0,1], \\ \frac{3(4-3v)}{2} \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{27}{17}$ if $P \in A \backslash (B \cup B' \cup B'')$.

Proof. The Zariski Decomposition in part a). follows from

$$-K_S - vA \sim_{\mathbb{R}} \Big(\frac{4}{3} - v\Big)A + \frac{1}{3}\Big(2B + B_1 + B_2 + B_3 + B_4 + B_5\Big).$$

A similar statement holds in other parts. We have $S_S(A) = \frac{17}{27}$. Thus, $\delta_P(S) \le \frac{27}{17}$ for $P \in A$. Note that for $P \in A \backslash B$ we have:

$$h(v) \le \begin{cases} \frac{(2+v)^2}{8} \text{ if } v \in [0,1], \\ \frac{3(4-3v)(8-5v)}{8} \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \le \frac{17}{27}$. Thus, $\delta_P(S) = \frac{27}{17}$. □

Remark 7.1.3. Minor additional computation shows that in cases **b), c)** we have $\delta_P(S) \ge \frac{35}{54}$ for all points $P \in A$.

Lemma 7.1.4. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

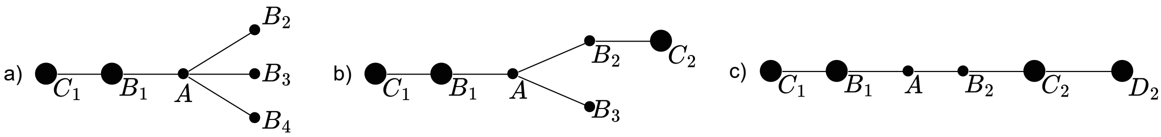


Figure 7.2: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{3}{2}$

Then $\tau(A) = \frac{3}{2}$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2B_1 + C_1) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{3}(2B_1 + C_1) - (v-1)(B_2 + B_3 + B_4) \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{3}(2B_1 + C_1) \text{ if } v \in [0,1], \\ \frac{v}{3}(2B_1 + C_1) + (v-1)(B_2 + B_3 + B_4) \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2B_1 + C_1) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{3}(2B_1 + C_1) - (v-1)(2B_2 + C_2 + B_3) \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{3}(2B_1 + C_1) \text{ if } v \in [0,1], \\ \frac{v}{3}(2B_1 + C_1) + (v-1)(2B_2 + C_2 + B_3) \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

c).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2B_1 + C_1) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{3}(2B_1 + C_1) - (v-1)(3B_2 + 2C_2 + D_2) \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{3}(2B_1 + C_1) \text{ if } v \in [0,1], \\ \frac{v}{3}(2B_1 + C_1) + (v-1)(3B_2 + 2C_2 + D_2) \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3 - 2v - \frac{v^2}{3} \text{ if } v \in [0,1], \\ \frac{2(3-2v)^2}{2} \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} 1 + \frac{v}{3} \text{ if } v \in [0,1], \\ 4(1 - \frac{2v}{3}) \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{3}{2}$ if $P \in A \backslash B_1$.

Proof. The Zariski Decomposition in part a). follows from

$$-K_S - vA \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)A + \frac{1}{2}\Big(2B_1 + C_1 + B_2 + B_3 + B_4\Big).$$

A similar statement holds in other parts. We have $S_S(A) = \frac{2}{3}$. Thus, $\delta_P(S) \le \frac{3}{2}$ for $P \in A$. Note that for $P \in A \backslash B_1$ we have:

$$h(v) \le \begin{cases} \frac{(3+v)^2}{18} \text{ if } v \in [0,1], \\ \frac{4(3-2v)(5v-3)}{9} \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \le \frac{2}{3}$. Thus, $\delta_P(S) = \frac{3}{2}$ if $P \in A\backslash B_1$. □

Lemma 7.1.5. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

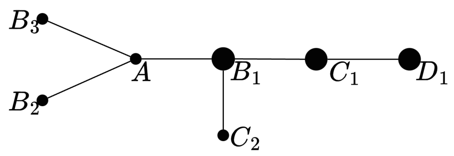


Figure 7.3: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{54}{37}$

Then $\tau(A) = \frac{3}{2}$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{4}(3B_1 + 2C_1 + D_1) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{4}(3B_1 + 2C_1 + D_1) - (v-1)(B_2 + B_3) \text{ if } v \in \left[1, \frac{4}{3}\right], \\ -K_S - vA - (v-1)(3B_1 + 2C_1 + D_1 + B_2 + B_3) - (3v-4)C_2 \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(3B_1 + 2C_1 + D_1) \text{ if } v \in [0,1], \\ \frac{v}{4}(3B_1 + 2C_1 + D_1) + (v-1)(B_2 + B_3) \text{ if } v \in \left[1, \frac{4}{3}\right], \\ (v-1)(3B_1 + 2C_1 + D_1 + B_2 + B_3) + (3v-4)C_2 \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3 - 2v - \frac{v^2}{4} \text{ if } v \in [0,1], \\ \frac{(10-7v)(2-v)}{4} \text{ if } v \in \left[1, \frac{4}{3}\right], \\ (3-2v)^2 \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} 1 + \frac{v}{4} \text{ if } v \in [0,1], \\ 3 - \frac{7v}{4} \text{ if } v \in \left[1, \frac{4}{3}\right], \\ 2(3-2v) \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{54}{37}$ if $P \in A\backslash B_1$.

Proof. The Zariski Decomposition follows from

$$-K_S - vA \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)A + \frac{1}{2}\Big(3B_1 + 2C_1 + D_1 + B_2 + B_3 + C_2\Big).$$

We have $S_S(A) = \frac{37}{54}$. Thus, $\delta_P(S) \le \frac{54}{37}$ for $P \in A$. Moreover, for $P \in A\backslash B_1$ we have:

$$h(v) \le \begin{cases} \frac{(v+4)^2}{32} \text{ if } v \in [0,1], \\ \frac{(12-7v)(v+4)}{32} \text{ if } v \in \left[1, \frac{4}{3}\right], \\ 2(3-2v)(2-v) \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \le \frac{7}{12} \le \frac{37}{54}$. Thus, $\delta_P(S) = \frac{54}{37}$ if $P \in A\backslash B_1$. □

Lemma 7.1.6. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

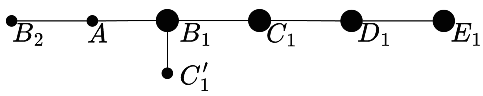


Figure 7.4: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{36}{25}$

Then $\tau(A) = \frac{3}{2}$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) - (v-1)B_2 \text{ if } v \in \left[1, \frac{5}{4}\right] \\ -K_S - vA - (v-1)(4B_1 + 3C_1 + 2D_1 + E_1) - (v-1)B_2 - (4v-5)C_1' \text{ if } v \in \left[\frac{5}{4}, \frac{3}{2}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) \text{ if } v \in [0,1], \\ \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) + (v-1)B_2 \text{ if } v \in \left[1, \frac{5}{4}\right], \\ (v-1)(4B_1 + 3C_1 + 2D_1 + E_1) + (v-1)B_2 + (4v-5)C_1' \text{ if } v \in \left[\frac{5}{4}, \frac{3}{2}\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3 - 2v - \frac{v^2}{5} \text{ if } v \in [0,1], \\ \frac{4v^2}{5} - 4v + 4 \text{ if } v \in \left[1, \frac{5}{4}\right], \\ (3-2v)^2 \text{ if } v \in \left[\frac{5}{4}, \frac{3}{2}\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} 1 + \frac{v}{5} \text{ if } v \in [0,1], \\ 2(1 - \frac{2v}{5}) \text{ if } v \in \left[1, \frac{5}{4}\right], \\ 2(3-2v) \text{ if } v \in \left[\frac{5}{4}, \frac{3}{2}\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{36}{25}$ if $P \in A \backslash B_1$.

Proof. The Zariski Decomposition follows from

$$-K_S - vA \sim_{\mathbb{R}} \left(\frac{3}{2} - v\right)A + \frac{1}{2}\left(4B_1 + 3C_1 + 2D_1 + E_1 + 2C_1' + B_2\right).$$

We have $S_S(A) = \frac{25}{36}$. Thus, $\delta_P(S) \leq \frac{36}{25}$ for $P \in A$. Moreover, for $P \in A \backslash B_1$ we have:

$$h(v) \leq \begin{cases} \frac{(v+5)^2}{50} \text{ if } v \in [0,1], \\ \frac{6v(5-2v)}{25} \text{ if } v \in \left[1, \frac{5}{4}\right], \\ 2(3-2v)(2-v) \text{ if } v \in \left[\frac{5}{4}, \frac{3}{2}\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \leq \frac{7}{12} < \frac{25}{36}$. Thus, $\delta_P(S) = \frac{36}{25}$ if $P \in A \backslash B_1$. □

Lemma 7.1.7. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

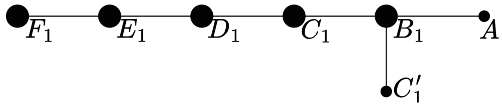


Figure 7.5: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{10}{7}$

Then $\tau(A) = \frac{3}{2}$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{6}(5B_1 + 4C_1 + 3D_1 + 2E_1 + F_1) \text{ if } v \in \left[0, \frac{6}{5}\right], \\ -K_S - vA - (v-1)(5B_1 + 4C_1 + 3D_1 + 2E_1 + F_1) - (5v-6)C_1' \text{ if } v \in \left[\frac{6}{5}, \frac{3}{2}\right], \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(5B_1 + 4C_1 + 3D_1 + 2E_1 + F_1) \text{ if } v \in \left[0, \frac{6}{5}\right], \\ (v-1)(5B_1 + 4C_1 + 3D_1 + 2E_1 + F_1) + (5v-6)C_1' \text{ if } v \in \left[\frac{6}{5}, \frac{3}{2}\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3 - 2v - \frac{v^2}{6} \text{ if } v \in \left[0, \frac{6}{5}\right], \\ (3-2v)^2 \text{ if } v \in \left[\frac{6}{5}, \frac{3}{2}\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} 1 + \frac{v}{6} \text{ if } v \in \left[0, \frac{6}{5}\right], \\ 2(3-2v) \text{ if } v \in \left[\frac{6}{5}, \frac{3}{2}\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{10}{7}$ if $P \in A \backslash B_1$.

Proof. The Zariski Decomposition follows from

$$-K_S - vA \sim_{\mathbb{R}} \left(\frac{3}{2} - v\right)A + \frac{1}{2}\left(6B_1 + 5C_1 + 4D_1 + 3E_1 + 2F_1 + 4C_1'\right).$$

We have $S_S(A) = \frac{7}{10}$. Thus, $\delta_P(S) \leq \frac{10}{7}$ for $P \in A$. Moreover, for $P \in A \backslash B_1$ we have:

$$h(v) \leq \begin{cases} \frac{(6+v)^2}{72} \text{ if } v \in \left[0, \frac{6}{5}\right], \\ 2(3-2v)^2 \text{ if } v \in \left[\frac{6}{5}, \frac{3}{2}\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \leq \frac{8}{15} < \frac{7}{10}$. Thus, $\delta_P(S) = \frac{10}{7}$ if $P \in A \backslash B_1$. □

Lemma 7.1.8. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

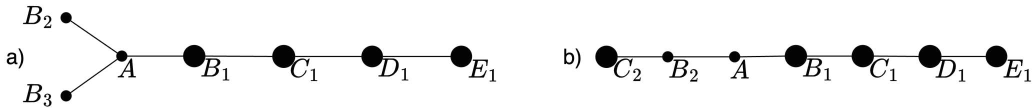


Figure 7.6: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{27}{19}$

Then $\tau(A)=\frac{5}{3}$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

a). $$P(v)=\begin{cases}-K_S-vA-\frac{v}{5}(4B_1+3C_1+2D_1+E_1) \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{5}(4B_1+3C_1+2D_1+E_1)-(v-1)(B_2+B_3) \text{ if } v\in\left[1,\frac{5}{3}\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{5}(4B_1+3C_1+2D_1+E_1) \text{ if } v\in[0,1],\\ \frac{v}{5}(4B_1+3C_1+2D_1+E_1)+(v-1)(B_2+B_3) \text{ if } v\in\left[1,\frac{5}{3}\right].\end{cases}$$

b). $$P(v)=\begin{cases}-K_S-vA-\frac{v}{5}(4B_1+3C_1+2D_1+E_1) \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{5}(4B_1+3C_1+2D_1+E_1)-(v-1)(2B_2+C_2) \text{ if } v\in\left[1,\frac{5}{3}\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{5}(4B_1+3C_1+2D_1+E_1) \text{ if } v\in[0,1],\\ \frac{v}{5}(4B_1+3C_1+2D_1+E_1)+(v-1)(2B_2+C_2) \text{ if } v\in\left[1,\frac{5}{3}\right].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}3-2v-\frac{v^2}{5} \text{ if } v\in[0,1],\\ \frac{(5-3v)^2}{5} \text{ if } v\in\left[1,\frac{5}{3}\right].\end{cases}\qquad P(v)\cdot A=\begin{cases}1+\frac{v}{5} \text{ if } v\in[0,1],\\ 3-\frac{9v}{5} \text{ if } v\in\left[1,\frac{5}{3}\right].\end{cases}$$

In this case $\delta_P(S)=\frac{27}{19}$ if $P\in A\backslash B_1$.

Proof. The Zariski Decomposition in part a). follows from

$$-K_S-vA\sim_{\mathbb{R}}\Big(\frac{5}{3}-v\Big)A+\frac{1}{3}\Big(4B_1+3C_1+2D_1+E_1+2B_2+2B_3\Big).$$

A similar statement holds in other parts. We have $S_S(A)=\frac{19}{27}$. Thus, $\delta_P(S)\leq\frac{27}{19}$ for $P\in A$. Moreover, for $P\in A\backslash B_1$ we have:

$$h(v)\leq\begin{cases}\frac{(v+5)^2}{50} \text{ if } v\in[0,1],\\ \frac{3(5-3v)(11v-5)}{50} \text{ if } v\in\left[1,\frac{5}{3}\right].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\leq\frac{17}{27}<\frac{19}{27}$. Thus, $\delta_P(S)=\frac{27}{19}$ if $P\in A\backslash B_1$. □

Lemma 7.1.9. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

Then $\tau(A)=2$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

a). $$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}(B_1+B_1') \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{2}(B_1+B_1')-(v-1)B_2 \text{ if } v\in[1,2].\end{cases}$$

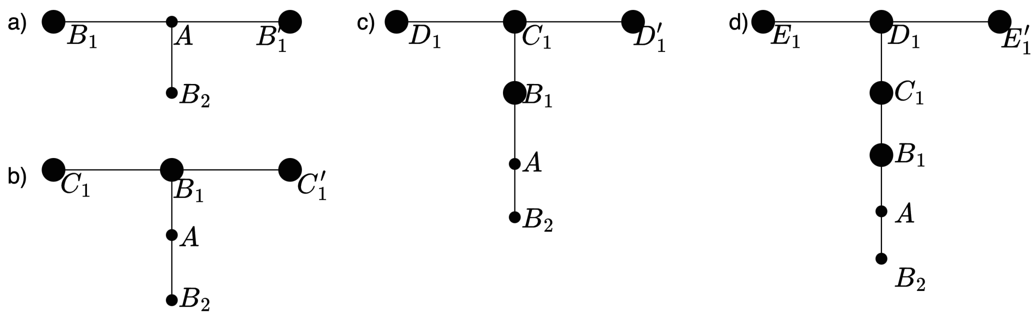


Figure 7.7: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{9}{7}$

$$N(v) = \begin{cases} \frac{v}{2}(B_1 + B_1') \text{ if } v \in [0,1], \\ \frac{v}{2}(B_1 + B_1') + (v-1)B_2 \text{ if } v \in [1,2]. \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}(2B_1 + C_1 + C_1') \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}(2B_1 + C_1 + C_1') - (v-1)B_2 \text{ if } v \in [1,2]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(2B_1 + C_1 + C_1') \text{ if } v \in [0,1], \\ \frac{v}{2}(2B_1 + C_1 + C_1') + (v-1)B_2 \text{ if } v \in [1,2]. \end{cases}$$

c).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}(2B_1 + 2C_1 + D_1 + D_1') \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}(2B_1 + 2C_1 + D_1 + D_1') - (v-1)B_2 \text{ if } v \in [1,2]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(2B_1 + 2C_1 + D_1 + D_1') \text{ if } v \in [0,1], \\ \frac{v}{2}(2B_1 + 2C_1 + D_1 + D_1') + (v-1)B_2 \text{ if } v \in [1,2]. \end{cases}$$

d).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}(2B_1 + 2C_1 + 2D_1 + E_1 + E_1') \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}(2B_1 + 2C_1 + 2D_1 + E_1 + E_1') - (v-1)B_2 \text{ if } v \in [1,2]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(2B_1 + 2C_1 + 2D_1 + E_1 + E_1') \text{ if } v \in [0,1], \\ \frac{v}{2}(2B_1 + 2C_1 + 2D_1 + E_1 + E_1') + (v-1)B_2 \text{ if } v \in [1,2]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3 - 2v \text{ if } v \in [0,1], \\ (2-v)^2 \text{ if } v \in [1,2]. \end{cases} \qquad P(v) \cdot A = \begin{cases} 1 \text{ if } v \in [0,1], \\ 2 - v \text{ if } v \in [1,2]. \end{cases}$$

In this case $\delta_P(S) = \frac{9}{7}$ *if* $P \in A \backslash (B_1 \cup B_1')$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + B_1 + B_1' + B_2$. A similar statement holds in other parts. We have $S_S(A) = \frac{7}{9}$. Thus, $\delta_P(S) \le \frac{9}{7}$ for $P \in A$. Note that for P as above we have:

$$h(v) \le \begin{cases} \frac{1}{2} \text{ if } v \in [0,1], \\ \frac{(2-v)v}{2} \text{ if } v \in [1,2]. \end{cases}$$

So $S(W^{E_1}_{\bullet,\bullet};P) \le \frac{5}{9} \le \frac{7}{9}$. Thus, $\delta_P(S) = \frac{9}{7}$. □

Lemma 7.1.10. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

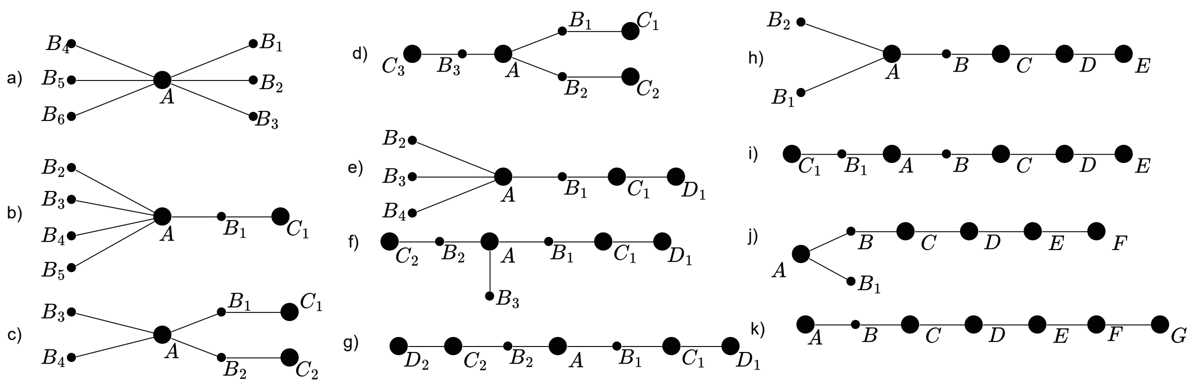


Figure 7.8: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{6}{5}$ with $\tau(A) = \frac{3}{2}$

Then $\tau(A) = \frac{3}{2}$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).
$$P(v) = \begin{cases} -K_S - vA \text{ if } v \in [0,1], \\ -K_S - vA - (v-1)(B_1+B_2+B_3+B_4+B_5+B_6) \text{ if } v \in \left[1,\frac{3}{2}\right]. \end{cases}$$
$$N(v) = \begin{cases} 0 \text{ if } v \in [0,1], \\ (v-1)(B_1+B_2+B_3+B_4+B_5+B_6) \text{ if } v \in \left[1,\frac{3}{2}\right]. \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vA \text{ if } v \in [0,1], \\ -K_S - vA - (v-1)(2B_1+C_1+B_2+B_3+B_4+B_5) \text{ if } v \in \left[1,\frac{3}{2}\right]. \end{cases}$$
$$N(v) = \begin{cases} 0 \text{ if } v \in [0,1], \\ (v-1)(2B_1+C_1+B_2+B_3+B_4+B_5) \text{ if } v \in \left[1,\frac{3}{2}\right]. \end{cases}$$

c).
$$P(v) = \begin{cases} -K_S - vA \text{ if } v \in [0,1], \\ -K_S - vA - (v-1)(2B_1+C_1+2B_2+C_2+B_3+B_4) \text{ if } v \in \left[1,\frac{3}{2}\right]. \end{cases}$$
$$N(v) = \begin{cases} 0 \text{ if } v \in [0,1], \\ (v-1)(2B_1+C_1+2B_2+C_2+B_3+B_4) \text{ if } v \in \left[1,\frac{3}{2}\right]. \end{cases}$$

d).
$$P(v)=\begin{cases}-K_S-vA \textit{ if } v\in[0,1],\\ -K_S-vA-(v-1)(2B_1+C_1+2B_2+C_2+2B_3+C_3) \textit{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$
$$N(v)=\begin{cases}0 \textit{ if } v\in[0,1],\\ (v-1)(2B_1+C_1+2B_2+C_2+2B_3+C_3) \textit{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

e).
$$P(v)=\begin{cases}-K_S-vA \textit{ if } v\in[0,1],\\ -K_S-vA-(v-1)(3B_1+2C_1+D_1+B_2+B_3+B_4) \textit{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$
$$N(v)=\begin{cases}0 \textit{ if } v\in[0,1],\\ (v-1)(3B_1+2C_1+D_1+B_2+B_3+B_4) \textit{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

f).
$$P(v)=\begin{cases}-K_S-vA \textit{ if } v\in[0,1],\\ -K_S-vA-(v-1)(3B_1+2C_1+D_1+2B_2+C_2+B_3) \textit{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$
$$N(v)=\begin{cases}0 \textit{ if } v\in[0,1],\\ (v-1)(3B_1+2C_1+D_1+2B_2+C_2+B_3) \textit{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

g).
$$P(v)=\begin{cases}-K_S-vA \textit{ if } v\in[0,1],\\ -K_S-vA-(v-1)(3B_1+2C_1+D_1+3B_2+2C_2+D_2) \textit{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$
$$N(v)=\begin{cases}0 \textit{ if } v\in[0,1],\\ (v-1)(3B_1+2C_1+D_1+3B_2+2C_2+D_2) \textit{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

h).
$$P(v)=\begin{cases}-K_S-vA \textit{ if } v\in[0,1],\\ -K_S-vA-(v-1)(4B+3C+2D+E+B_1+B_2) \textit{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$
$$N(v)=\begin{cases}0 \textit{ if } v\in[0,1],\\ (v-1)(4B+3C+2D+E+B_1+B_2) \textit{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

i).
$$P(v)=\begin{cases}-K_S-vA \textit{ if } v\in[0,1],\\ -K_S-vA-(v-1)(4B+3C+2D+E+2B_1+C_1) \textit{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$
$$N(v)=\begin{cases}0 \textit{ if } v\in[0,1],\\ (v-1)(4B+3C+2D+E+2B_1+C_1) \textit{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

j).
$$P(v)=\begin{cases}-K_S-vA \textit{ if } v\in[0,1],\\ -K_S-vA-(v-1)(5B+4C+3D+2E+F+B_1) \textit{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$
$$N(v)=\begin{cases}0 \textit{ if } v\in[0,1],\\ (v-1)(5B+4C+3D+2E+F+B_1) \textit{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

k).
$$P(v)=\begin{cases}-K_S-vA \textit{ if } v\in[0,1],\\ -K_S-vA-(v-1)(6B+5C+4D+3E+2F+G) \textit{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

$$N(v) = \begin{cases} 0 \textit{ if } v \in [0,1], \\ (v-1)(6B+5C+4D+3E+2F+G) \textit{ if } v \in \left[1,\frac{3}{2}\right]. \end{cases}$$

Moreover:

$$(P(v))^2 = \begin{cases} 3-2v^2 \textit{ if } v \in [0,1], \\ (3-2v)^2 \textit{ if } v \in \left[1,\frac{3}{2}\right]. \end{cases} \quad P(v)\cdot A = \begin{cases} 2v \textit{ if } v \in [0,1], \\ 2(3-2v) \textit{ if } v \in \left[1,\frac{3}{2}\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{6}{5}$ *if* $P \in A \backslash B$.

Proof. The Zariski Decomposition in part a). follows from

$$-K_S - vA \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)A + \frac{1}{2}\Big(B_1 + B_2 + B_3 + B_4 + B_5 + B_6\Big).$$

A similar statement holds in other parts. We have $S_S(A) = \frac{5}{6}$, Thus, $\delta_P(S) \le \frac{6}{5}$ for $P \in A$. Note that for $P \in A$ we have:

$$h(v) \le \begin{cases} 2v^2 \text{ if } v \in [0,1], \\ 2v(3-2v) \text{ if } v \in \left[1,\frac{3}{2}\right]. \end{cases}$$

So we have $S(W^{A}_{\bullet,\bullet};P) \le \frac{5}{6}$. Thus, $\delta_P(S) = \frac{6}{5}$ if $P \in A \backslash B$. □

Lemma 7.1.11. *Suppose* P *belongs to a* (-1)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

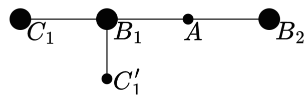


Figure 7.9: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{6}{5}$ with $\tau(A) = 2$

Then $\tau(A) = 2$ *and the Zariski Decomposition of the divisor* $-K_S - vA$ *is given by:*

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2B_1+C_1) - \frac{v}{2}B_2 \textit{ if } v \in \left[0,\frac{3}{2}\right], \\ -K_S - vA - (v-1)(2B_1+C_1) - \frac{v}{2}B_2 - (2v-3)C_1' \textit{ if } v \in \left[\frac{3}{2},2\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2B_1+C_1) + \frac{v}{2}B_2 \textit{ if } v \in \left[0,\frac{3}{2}\right], \\ (v-1)(2B_1+C_1) + \frac{v}{2}B_2 + (2v-3)C_1' \textit{ if } v \in \left[\frac{3}{2},2\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3-2v+\frac{v^2}{6} \textit{ if } v \in \left[0,\frac{3}{2}\right], \\ \frac{3(2-v)^2}{2} \textit{ if } v \in \left[\frac{3}{2},2\right]. \end{cases} \quad P(v)\cdot A = \begin{cases} 1-\frac{v}{6} \textit{ if } v \in \left[0,\frac{3}{2}\right], \\ 3-\frac{3v}{2} \textit{ if } v \in \left[\frac{3}{2},2\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{6}{5}$ *if* $P \in A\backslash(B_1 \cup B_2)$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + 2B_1 + C_1 + B_2 + C_1'$. We have $S_S(A) = \frac{5}{6}$. Thus, $\delta_P(S) \leq \frac{6}{5}$ for $P \in A$. Note that for $P \in A\backslash(B_1 \cup B_2)$ we have:

$$h(v) = \begin{cases} \frac{(6-v)^2}{72} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{9(2-v)^2}{8} \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \leq \frac{5}{6}$. Thus, $\delta_P(S) = \frac{6}{5}$ if $P \in A\backslash(B_1 \cup B_2)$. □

Lemma 7.1.12. *Suppose* P *belongs to a* (-1)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

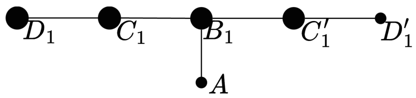


Figure 7.10: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{27}{23}$

Then $\tau(A) = 2$ *and the Zariski Decomposition of the divisor* $-K_S - vA$ *is given by:*

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{5}(6B_1 + 3C_1' + 4C_1 + 2D_1) \text{ if } v \in \left[0, \frac{5}{3}\right], \\ -K_S - vA - (v-1)(3B_1 + 2C_1 + D_1) - (3v-4)C_1' - (3v-5)D_1' \text{ if } v \in \left[\frac{5}{3}, 2\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(6B_1 + 3C_1' + 4C_1 + 2D_1) \text{ if } v \in \left[0, \frac{5}{3}\right], \\ (v-1)(3B_1 + 2C_1 + D_1) + (3v-4)C_1' + (3v-5)D_1' \text{ if } v \in \left[\frac{5}{3}, 2\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} \frac{v^2}{5} - 2v + 3 \text{ if } v \in \left[0, \frac{5}{3}\right], \\ 2(2-v)^2 \text{ if } v \in \left[\frac{5}{3}, 2\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} 1 - \frac{v}{5} \text{ if } v \in \left[0, \frac{5}{3}\right], \\ 2(2-v) \text{ if } v \in \left[\frac{5}{3}, 2\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{27}{23}$ *if* $P \in A\backslash B_1$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + 3B_1 + 2C_1 + D_1 + 2C_1' + D_1'$. We have $S_S(A) = \frac{23}{27}$. Thus, $\delta_P(S) \leq \frac{27}{23}$ for $P \in A$. Moreover, for $P \in A\backslash B_1$ we have:

$$h(v) = \begin{cases} \frac{(5-v)^2}{50} \text{ if } v \in \left[0, \frac{5}{3}\right], \\ 2(2-v)^2 \text{ if } v \in \left[\frac{5}{3}, 2\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \leq \frac{11}{27} < \frac{23}{27}$. Thus, $\delta_P(S) = \frac{27}{23}$ if $P \in A\backslash B_1$. □

Lemma 7.1.13. *Suppose* P *belongs to a* (-1)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

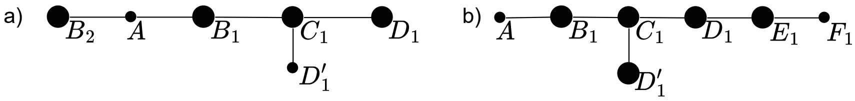


Figure 7.11: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{9}{8}$

Then the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).
$$P(v) = -K_S - vA - \frac{v}{4}(3B_1 + 2C_1 + D_1 + 2B_2) \text{ if } v \in [0,2].$$
$$N(v) = \frac{v}{4}(3B_1 + 2C_1 + D_1 + 2B_2) \text{ if } v \in [0,2].$$

b).
$$P(v) = -K_S - vA - \frac{v}{4}(5B_1 + 6C_1 + 3D'_1 + 4D_1 + 2E_1) \text{ if } v \in [0,2].$$
$$N(v) = \frac{v}{4}(5B_1 + 6C_1 + 3D'_1 + 4D_1 + 2E_1) \text{ if } v \in [0,2].$$

Moreover,

$$(P(v))^2 = \frac{(2-v)(6-v)}{4} \text{ if } v \in [0,2] \text{ and } P(v) \cdot A = 1 - \frac{v}{4} \text{ if } v \in [0,2].$$

In this case $\delta_P(S) = \frac{9}{8}$ if $P \in A \backslash B_1$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + 2B_1 + 2C_1 + D_1 + D'_1 + B_2$. A similar statement holds in other parts. We have $S_S(A) = \frac{8}{9}$. Thus, $\delta_P(S) \leq \frac{9}{8}$ for $P \in A$. Moreover, for $P \in A \backslash B_1$ we have $h(v) = \frac{(4-v)(3v+4)}{32}$ if $v \in [0,2]$. So $S(W^A_{\bullet,\bullet};P) \leq \frac{5}{6} < \frac{8}{9}$. Thus, $\delta_P(S) = \frac{9}{8}$ if $P \in A \backslash B_1$. □

Lemma 7.1.14. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

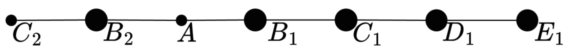


Figure 7.12: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{18}{17}$

Then $\tau(A) = \frac{5}{2}$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B_2 - \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) \text{ if } v \in [0,2], \\ -K_S - vA - (v-1)B_2 - (v-2)C_2 - \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{2}B_2 + \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) \text{ if } v \in [0,2], \\ (v-1)B_2 + (v-2)C_2 + \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} \frac{3v^2}{10} - 2v + 3 \text{ if } v \in [0,2], \\ \frac{(5-2v)^2}{5} \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases} \quad P(v)\cdot A = \begin{cases} 1 - \frac{3v}{10} \text{ if } v \in [0,2], \\ 2(1-\frac{2v}{5}) \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{18}{17}$ *if* $P \in A \backslash B_1$.

Proof. The Zariski Decomposition follows from

$$-K_S - vA \sim_{\mathbb{R}} \Big(\frac{5}{2} - v\Big)A + \frac{1}{2}\Big(4B_1 + 3C_1 + 2D_1 + E_1 + 3B_2 + C_2\Big).$$

We have $S_S(A) = \frac{17}{18}$. Thus, $\delta_P(S) \leq \frac{18}{17}$ for $P \in A$. Moreover, for $P \in A \backslash B_1$ we have:

$$h(v) = \begin{cases} \frac{(10-3v)(7v+10)}{200} \text{ if } v \in [0,2], \\ \frac{6v(5-2v)}{25} \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{5}{6} < \frac{17}{18}$. Thus, $\delta_P(S) = \frac{18}{17}$ if $P \in A \backslash B_1$. □

Lemma 7.1.15. *Suppose* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

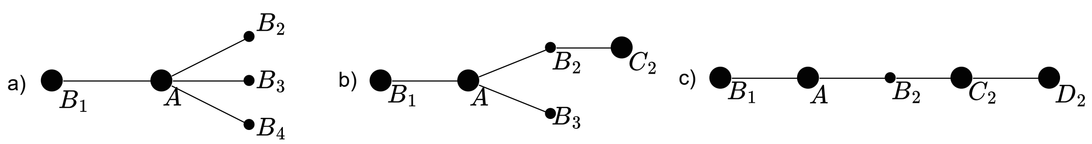


Figure 7.13: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = 1$ with $\tau(A) = 2$

Then $\tau(A) = 2$ *and the Zariski Decomposition of the divisor* $-K_S - vA$ *is given by:*

a).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B_1 - (v-1)(B_2 + B_3 + B_4) \text{ if } v \in [1,2]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ \frac{v}{2}B_1 + (v-1)(B_2 + B_3 + B_4) \text{ if } v \in [1,2]. \end{cases}$$

b).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B_1 - (v-1)(2B_2 + C_2 + B_3) \text{ if } v \in [1,2]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ \frac{v}{2}B_1 + (v-1)(2B_2 + C_2 + B_3) \text{ if } v \in [1,2]. \end{cases}$$

c).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B_1 - (v-1)(3B_2 + 2C_2 + D_2) \text{ if } v \in [1,2]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ \frac{v}{2}B_1 + (v-1)(3B_2+2C_2+D_2) \text{ if } v \in [1,2]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3 - \frac{3v^2}{2} \text{ if } v \in [0,1], \\ \frac{3(2-v)^2}{2} \text{ if } v \in [1,2]. \end{cases} \quad P(v)\cdot A = \begin{cases} \frac{3v}{2} \text{ if } v \in [0,1], \\ 3 - \frac{3v}{2} \text{ if } v \in [1,2]. \end{cases}$$

In this case $\delta_P(S) = 1$ *if* $P \in A$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + B_1 + B_2 + B_3 + B_4$. A similar statement holds in other parts. We have $S_S(A) = 1$. Thus, $\delta_P(S) \leq 1$ for $P \in A$. Moreover if $P \in A \cap B_1$ or if $P \in A \backslash B_1$:

$$h(v) \leq \begin{cases} \frac{15v^2}{8} \text{ if } v \in [0,1], \\ \frac{3(2-v)(6-v)}{8} \text{ if } v \in [1,2]. \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{15v^2}{8} \text{ if } v \in [0,1], \\ \frac{9(2-v)(3v-2)}{8} \text{ if } v \in [1,2]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq 1$. Thus, $\delta_P(S) = 1$ if $P \in A$. □

Lemma 7.1.16. *Suppose* P *belongs to a* (-1)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

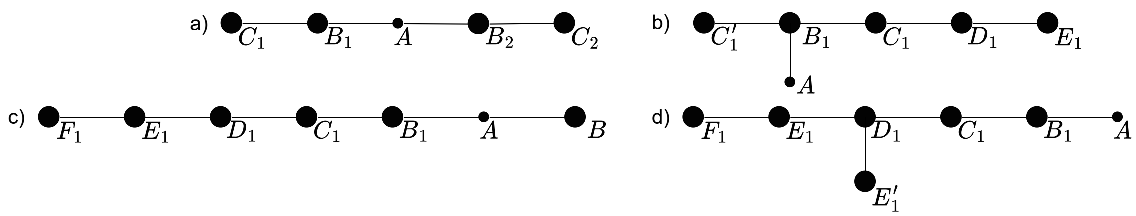


Figure 7.14: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = 1$ with $\tau(A) = 3$

Then $\tau(A) = 3$ *and the Zariski Decomposition of the divisor* $-K_S - vA$ *is given by:*

a).
$$P(v) = -K_S - vA - \frac{v}{3}(2B_1 + C_1 + 2B_2 + C_2) \text{ if } v \in [0,3].$$
$$N(v) = \frac{v}{3}(2B_1 + C_1 + 2B_2 + C_2) \text{ if } v \in [0,3].$$

b).
$$P(v) = -K_S - vA - \frac{v}{6}(4B_1 + 3C_1 + 2D_1 + E_1 + 2C_1') \text{ if } v \in [0,3].$$
$$N(v) = \frac{v}{6}(4B_1 + 3C_1 + 2D_1 + E_1 + 2C_1') \text{ if } v \in [0,3].$$

c).
$$P(v) = -K_S - vA - \frac{v}{6}(5B_1 + 4C_1 + 3D_1 + 2E_1 + F_1 + 3B) \text{ if } v \in [0,3].$$
$$N(v) = \frac{v}{6}(5B_1 + 4C_1 + 3D_1 + 2E_1 + F_1 + 3B) \text{ if } v \in [0,3].$$

d).
$$P(v) = -K_S - vA - \frac{v}{6}(2F_1 + 4E_1 + 6D_1 + 5C_1 + 4B_1 + 3E_1') \text{ if } v \in [0,3].$$

$$N(v) = \frac{v}{6}(2F_1 + 4E_1 + 6D_1 + 5C_1 + 4B_1 + 3E_1') \text{ if } v \in [0,3].$$

Moreover,

$$(P(v))^2 = \frac{(3-v)^2}{2} \text{ if } v \in [0,3] \text{ and } P(v)\cdot A = 1 - \frac{v}{3} \text{ if } v \in [0,3].$$

In this case $\delta_P(S) = 1$ if $P \in A\backslash(B_1 \cup B_2)$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} (3-v)A + 2B_1 + C_1 + 2B_2 + C_2$. A similar statement holds in other parts. We have $S_S(A) = 1$. Thus, $\delta_P(S) \leq 1$ for $P \in A$. Moreover, for $P \in A\backslash(B_1 \cup B_2)$ we have $h(v) = \frac{(3-v)(2v+3)}{18}$ if $v \in [0,3]$. So $S(W^A_{\bullet,\bullet};P) \leq \frac{5}{6} < 1$. Thus, $\delta_P(S) = 1$ if $P \in A\backslash(B_1 \cup B_2)$. ☐

Lemma 7.1.17. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

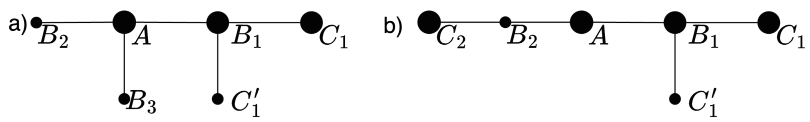


Figure 7.15: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{18}{19}$

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2B_1 + C_1) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{3}(2B_1 + C_1) - (v-1)(B_2 + B_3) \text{ if } v \in \left[1, \frac{3}{2}\right], \\ -K_S - vA - (v-1)(2B_1 + C_1 + B_2 + B_3) - (2v-3)C_1' \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2B_1 + C_1) \text{ if } v \in [0,1], \\ \frac{v}{3}(2B_1 + C_1) + (v-1)(B_2 + B_3) \text{ if } v \in \left[1, \frac{3}{2}\right], \\ (v-1)(2B_1 + C_1 + B_2 + B_3) + (2v-3)C_1' \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2B_1 + C_1) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{3}(2B_1 + C_1) - (v-1)(2B_2 + C_2) \text{ if } v \in \left[1, \frac{3}{2}\right], \\ -K_S - vA - (v-1)(2B_1 + C_1 + 2B_2 + C_2) - (2v-3)C_1' \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2B_1 + C_1) \text{ if } v \in [0,1], \\ \frac{v}{3}(2B_1 + C_1) + (v-1)(2B_2 + C_2) \text{ if } v \in \left[1, \frac{3}{2}\right], \\ (v-1)(2B_1 + C_1 + 2B_2 + C_2) + (2v-3)C_1' \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} \frac{(3-2v)(3+2v)}{3} \text{ if } v \in [0,1], \\ \frac{2v^2}{3} - 4v + 5 \text{ if } v \in \left[1, \frac{3}{2}\right], \\ 2(2-v)^2 \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{4}{3}v \text{ if } v \in [0,1], \\ 2(1-\frac{v}{3}) \text{ if } v \in \left[1, \frac{3}{2}\right], \\ 2(2-v) \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{18}{19}$ *if* $P \in A \backslash B_1$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + 2B_1 + C_1 + C_1' + B_2 + B_3$. A similar statement holds in other parts. We have $S_S(A) = \frac{19}{18}$. Thus, $\delta_P(S) \leq \frac{18}{19}$ for $P \in A$. Moreover, for $P \in A \backslash B_1$ we have:

$$h(v) \leq \begin{cases} \frac{8v^2}{9} \text{ if } v \in [0,1], \\ \frac{4v(3-v)}{9} \text{ if } v \in \left[1, \frac{3}{2}\right], \\ 2v(2-v) \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \leq \frac{8}{9} \leq \frac{19}{18}$. Thus, $\delta_P(S) = \frac{18}{19}$ if $P \in A \backslash B_1$. □

Lemma 7.1.18. *Suppose* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

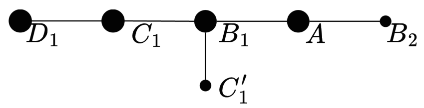


Figure 7.16: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{27}{29}$

Then $\tau(A) = 2$ *and the Zariski Decomposition of the divisor* $-K_S - vA$ *is given by:*

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{4}(3B_1 + 2C_1 + D_1) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{4}(3B_1 + 2C_1 + D_1) - (v-1)B_2 \text{ if } v \in \left[1, \frac{4}{3}\right]. \\ -K_S - vA - (v-1)(3B_1 + 2C_1 + D_1 + B_2) - (3v-4)C_1' \text{ if } v \in \left[\frac{4}{3}, 2\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(3B_1 + 2C_1 + D_1) \text{ if } v \in [0,1], \\ \frac{v}{4}(3B_1 + 2C_1 + D_1) + (v-1)B_2 \text{ if } v \in \left[1, \frac{4}{3}\right], \\ (v-1)(3B_1 + 2C_1 + D_1 + B_2) + (3v-4)C_1' \text{ if } v \in \left[\frac{4}{3}, 2\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3 - \frac{5v^2}{4} \text{ if } v \in [0,1], \\ 4 - 2v - \frac{v^2}{4} \text{ if } v \in \left[1, \frac{4}{3}\right], \\ 2(2-v)^2 \text{ if } v \in \left[\frac{4}{3}, 2\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{5v}{4} \text{ if } v \in [0,1], \\ 1 + \frac{v}{4} \text{ if } v \in \left[1, \frac{4}{3}\right], \\ 2(2-v) \text{ if } v \in \left[\frac{4}{3}, 2\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{27}{29}$ *if* $P \in A \backslash B_1$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + 3B_1 + 2C_1 + D_1 + 2C_1' + B_2$. We have $S_S(A) = \frac{29}{27}$. Thus, $\delta_P(S) \leq \frac{27}{29}$ for $P \in A$. Moreover, for $P \in A \backslash B_1$ we have:

$$h(v) \leq \begin{cases} \frac{25v^2}{32} \text{ if } v \in [0,1], \\ \frac{(v+4)(9v-4)}{32} \text{ if } v \in \left[0, \frac{4}{3}\right], \\ 2(2-v) \text{ if } v \in \left[\frac{4}{3}, 2\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{19}{27} < \frac{29}{27}$. Thus, $\delta_P(S) = \frac{27}{29}$ if $P \in A \backslash B_1$. □

Lemma 7.1.19. *Suppose* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

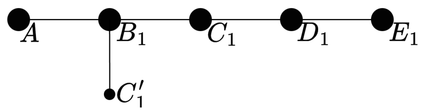


Figure 7.17: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{12}{13}$

Then $\tau(A) = 2$ *and the Zariski Decomposition of the divisor* $-K_S - vA$ *is given by:*

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) \text{ if } v \in \left[0, \frac{5}{4}\right], \\ -K_S - vA - (v-1)(4B_1 + 3C_1 + 2D_1 + E_1) - (4v-5)C_1' \text{ if } v \in \left[\frac{5}{4}, 2\right], \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) \text{ if } v \in \left[0, \frac{5}{4}\right], \\ (v-1)(4B_1 + 3C_1 + 2D_1 + E_1) + (4v-5)C_1' \text{ if } v \in \left[\frac{5}{4}, 2\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3 - \frac{6v^2}{5} \text{ if } v \in \left[0, \frac{5}{4}\right], \\ 2(2-v)^2 \text{ if } v \in \left[\frac{5}{4}, 2\right], \end{cases} \qquad P(v) \cdot A = \begin{cases} \frac{6v}{5} \text{ if } v \in \left[0, \frac{5}{4}\right], \\ 2(2-v) \text{ if } v \in \left[\frac{5}{4}, 2\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{12}{13}$ *if* $P \in A \backslash B_1$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + 4B_1 + 3C_1 + 2D_1 + E_1 + 3C_1'$. We have $S_S(A) = \frac{13}{12}$. Thus, $\delta_P(S) \leq \frac{12}{13}$ for $P \in A$. Moreover, for $P \in A \backslash B_1$ we have:

$$h(v) \leq \begin{cases} \frac{18v^2}{25} \text{ if } v \in \left[0, \frac{5}{4}\right], \\ 2(2-v)^2 \text{ if } v \in \left[\frac{5}{4}, 2\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{1}{2} < \frac{13}{12}$. Thus, $\delta_P(S) = \frac{12}{13}$ if $P \in A \backslash B_1$. □

Lemma 7.1.20. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

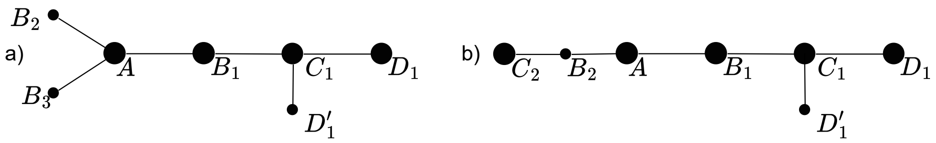


Figure 7.18: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{9}{10}$

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{4}(3B_1 + 2C_1 + D_1) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{4}(3B_1 + 2C_1 + D_1) - (v-1)(B_2 + B_3) \text{ if } v \in [1,2]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{4}(3B_1 + 2C_1 + D_1) \text{ if } v \in [0,1], \\ \frac{v}{4}(3B_1 + 2C_1 + D_1) + (v-1)(B_2 + B_3) \text{ if } v \in [1,2]. \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{4}(3B_1 + 2C_1 + D_1) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{4}(3B_1 + 2C_1 + D_1) - (v-1)(2B_2 + C_2) \text{ if } v \in [1,2]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{4}(3B_1 + 2C_1 + D_1) \text{ if } v \in [0,1], \\ \frac{v}{4}(3B_1 + 2C_1 + D_1) + (v-1)(2B_2 + C_2) \text{ if } v \in [1,2]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3 - \frac{5v^2}{4} \text{ if } v \in [0,1], \\ \frac{(2-v)(10-3v)}{2} \text{ if } v \in [1,2]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{5v}{4} \text{ if } v \in [0,1], \\ 2 - \frac{3v}{4} \text{ if } v \in [1,2]. \end{cases}$$

In this case $\delta_P(S) = \frac{9}{10}$ if $P \in A \backslash B_1$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + 2B_1 + 2C_1 + D_1 + D_1' + B_2 + B_3$. A similar statement holds in other parts. We have $S_S(A) = \frac{10}{9}$. Thus, $\delta_P(S) \le \frac{9}{10}$ for $P \in A$. Moreover, for $P \in A \backslash B_1$ we have:

$$h(v) \le \begin{cases} \frac{25v^2}{32} \text{ if } v \in [0,1], \\ \frac{(8-3v)(13v-8)}{32} \text{ if } v \in [1,2]. \end{cases}$$

So we have $S(W^A_{\bullet,\bullet}; P) \le \frac{17}{18} < \frac{10}{9}$. Thus, $\delta_P(S) = \frac{9}{10}$ if $P \in A \backslash B_1$. $\square$

Lemma 7.1.21. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

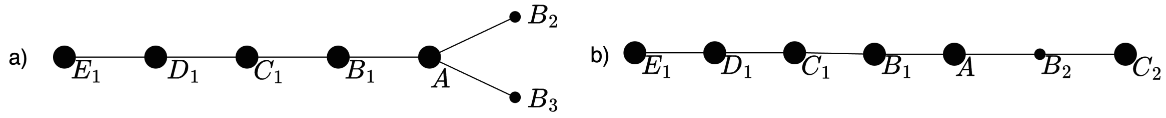


Figure 7.19: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{6}{7}$

Then $\tau(A) = \frac{5}{2}$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) - (v-1)(B_2 + B_3) \text{ if } v \in \left[1, \frac{5}{2}\right]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) \text{ if } v \in [0,1], \\ \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) + (v-1)(B_2 + B_3) \text{ if } v \in \left[1, \frac{5}{2}\right]. \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) - (v-1)(2B_2 + C_2) \text{ if } v \in \left[1, \frac{5}{2}\right]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) \text{ if } v \in [0,1], \\ \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) + (v-1)(2B_2 + C_2) \text{ if } v \in \left[1, \frac{5}{2}\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3 - \frac{6v^2}{5} \text{ if } v \in [0,1], \\ \frac{(5-2v)^2}{5} \text{ if } v \in \left[1, \frac{5}{2}\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{6v}{5} \text{ if } v \in [0,1], \\ 2(1 - \frac{2v}{5}) \text{ if } v \in \left[1, \frac{5}{2}\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{6}{7}$ if $P \in A \backslash B_1$.

Proof. The Zariski Decomposition in part a). follows from

$$-K_S - vA \sim_{\mathbb{R}} \Big(\frac{5}{2} - v\Big)A + \frac{1}{2}\Big(4B_1 + 3C_1 + 2D_1 + E_1 + 3B_2 + 3B_3\Big).$$

A similar statement holds in other parts. We have $S_S(A) = \frac{7}{6}$. Thus, $\delta_P(S) \le \frac{6}{7}$ for $P \in A$. Moreover, for $P \in A \backslash B_1$ we have:

$$h(v) \le \begin{cases} \frac{18v^2}{25} \text{ if } v \in [0,1], \\ \frac{6v(5-2v)}{25} \text{ if } v \in \left[1, \frac{5}{2}\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \le \frac{7}{10} < \frac{7}{6}$. Thus, $\delta_P(S) = \frac{6}{7}$ if $P \in A \backslash B_1$. □

Lemma 7.1.22. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

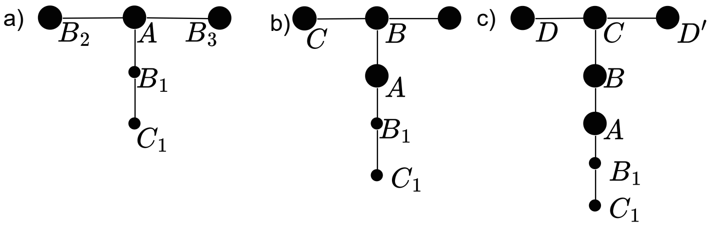


Figure 7.20: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{9}{11}$

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}(B_2+B_3) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}(B_2+B_3) - (v-1)B_1 \text{ if } v \in [1,2]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{2}(B_2+B_3) \text{ if } v \in [0,1], \\ \frac{v}{2}(B_2+B_3) + (v-1)B_1 \text{ if } v \in [1,2]. \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}(2B+C+C') \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}(2B+C+C') - (v-1)B_1 \text{ if } v \in [1,2]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{2}(2B+C+C') \text{ if } v \in [0,1], \\ \frac{v}{2}(2B+C+C') + (v-1)B_1 \text{ if } v \in [1,2]. \end{cases}$$

c).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}(2B+2C+D+D') \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}(2B+2C+D+D') - (v-1)B_1 \text{ if } v \in [1,2]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{2}(2B+2C+D+D') \text{ if } v \in [0,1], \\ \frac{v}{2}(2B+2C+D+D') + (v-1)B_1 \text{ if } v \in [1,2]. \end{cases}$$

Moreover,
$$(P(v))^2 = \begin{cases} 3 - v^2 \text{ if } v \in [0,1], \\ 4 - 2v \text{ if } v \in [1,2]. \end{cases} \quad P(v) \cdot A = \begin{cases} v \text{ if } v \in [0,1], \\ 1 \text{ if } v \in [1,2]. \end{cases}$$

In this case $\delta_P(S) = \frac{9}{11}$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + B_2 + B_3 + 2B_1 + C_1$. A similar statement holds in other parts. We have $S_S(A) = \frac{11}{9}$. Thus, $\delta_P(S) \leq \frac{9}{11}$ for $P \in A$. Moreover, for $P \in A \backslash B$ we have:

$$h(v) \leq \begin{cases} \frac{v^2}{2} \text{ if } v \in [0,1], \\ v - \frac{1}{2} \text{ if } v \in [1,2]. \end{cases} \quad \text{or } h(v) \leq \begin{cases} v^2 \text{ if } v \in [0,1], \\ \frac{(v+1)}{2} \text{ if } v \in [1,2]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{7}{9} < \frac{11}{9}$ or $S(W^A_{\bullet,\bullet};P) \leq \frac{19}{18} < \frac{11}{9}$. Thus, $\delta_P(S) = \frac{9}{11}$ if $P \in A$. □

Lemma 7.1.23. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

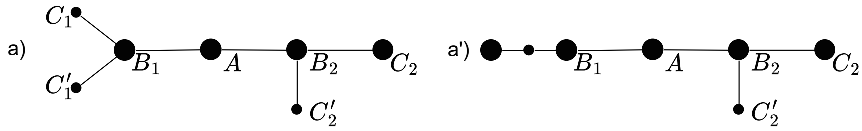


Figure 7.21: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{18}{23}$

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B_1 - \frac{v}{3}(2B_2 + C_2) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ -K_S - vA - \frac{v}{2}B_1 - (v-1)(2B_2 + C_2) - (2v-3)C_2' \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}B_1 + \frac{v}{3}(2B_2 + C_2) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{v}{2}B_1 + (v-1)(2B_2 + C_2) + (2v-3)C_2' \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3 - \frac{5v^2}{6} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{(2-v)(6-v)}{2} \text{ if } v \in \left[\frac{3}{2}, 2\right], \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{5v}{6} \text{ if } v \in \left[0, \frac{3}{2}\right]. \\ 2 - \frac{v}{2} \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{18}{23}$ if $P \in A \backslash B_2$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + 2B_1 + C_1 + C_1' + 2B_2 + C_2 + C_2'$. We have $S_S(A) = \frac{23}{18}$. Thus, $\delta_P(S) \leq \frac{18}{23}$ for $P \in A$. Moreover, for $P \in A \backslash B_2$ we have:

$$h(v) \leq \begin{cases} \frac{55v^2}{72} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{(4+v)(4-v)}{8} \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \leq \frac{10}{9} < \frac{23}{18}$. Thus, $\delta_P(S) = \frac{18}{23}$ if $P \in A \backslash B_2$. □

Lemma 7.1.24. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

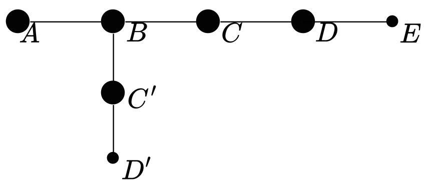


Figure 7.22: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{27}{35}$

Then $\tau(A)=2$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

$$P(v)=\begin{cases}-K_S-vA-\frac{v}{5}(6B+4C+2D+3C') \text{ if } v\in\left[0,\frac{5}{3}\right],\\ -K_S-vA-(v-1)(3B+2C+D)-(3v-4)C'-(3v-5)D' \text{ if } v\in\left[\frac{5}{3},2\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{5}(6B+4C+2D+3C') \text{ if } v\in\left[0,\frac{5}{3}\right],\\ (v-1)(3B+2C+D)+(3v-4)C'+(3v-5)D' \text{ if } v\in\left[\frac{5}{3},2\right].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}3-\frac{4v^2}{5} \text{ if } v\in\left[0,\frac{5}{3}\right],\\ (2-v)(4-v) \text{ if } v\in\left[\frac{5}{3},2\right].\end{cases}\qquad P(v)\cdot A=\begin{cases}\frac{4v}{5} \text{ if } v\in\left[0,\frac{5}{3}\right],\\ 3-v \text{ if } v\in\left[\frac{5}{3},2\right].\end{cases}$$

In this case $\delta_P(S)=\frac{27}{35}$ if $P\in A\backslash B$.

Proof. The Zariski Decomposition follows from $-K_S-vA\sim_{\mathbb{R}}(2-v)A+4B+3C+2D+E+3C'+2D'$. We have $S_S(A)=\frac{35}{27}$. Thus, $\delta_P(S)\le\frac{27}{35}$ for $P\in A$. Moreover, for $P\in A\backslash B$ we have:

$$h(v)\le\begin{cases}\frac{16v^2}{50} \text{ if } v\in\left[0,\frac{5}{3}\right],\\ \frac{(3-v)^2}{2} \text{ if } v\in\left[\frac{5}{3},2\right].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\le\frac{13}{27}<\frac{35}{27}$. Thus, $\delta_P(S)=\frac{27}{35}$ if $P\in A\backslash B$. □

Lemma 7.1.25. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

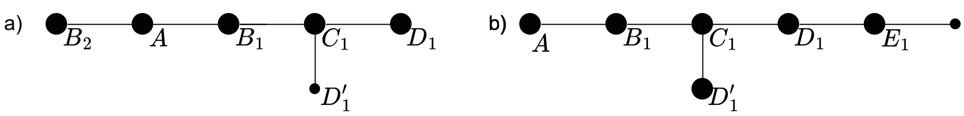


Figure 7.23: Dual graph: $(-K_S)^2=3$ and $\delta_P(S)=\frac{3}{4}$

Then $\tau(A)=2$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

a).
$$P(v)=-K_S-vA-\frac{v}{4}(2B_2+3B_1+2C_1+D_1) \text{ if } v\in[0,2].$$
$$N(v)=\frac{v}{4}(2B_2+3B_1+2C_1+D_1) \text{ if } v\in[0,2].$$

b).
$$P(v)=-K_S-vA-\frac{v}{4}(5B_1+6C_1+4D_1+2E_1+3D_1') \text{ if } v\in[0,2].$$
$$N(v)=\frac{v}{4}(5B_1+6C_1+4D_1+2E_1+3D_1') \text{ if } v\in[0,2].$$

Moreover,

$$(P(v))^2=\frac{3(2-v)(2+v)}{4} \text{ if } v\in[0,2] \text{ and } P(v)\cdot A=\frac{3v}{4} \text{ if } v\in[0,2].$$

In this case $\delta_P(S) = \frac{3}{4}$ *if* $P \in A \backslash B_1$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + B_2 + 3B_1 + 4C_1 + 2D_1 + 3D_1'$. A similar statement holds in other parts. We have $S_S(A) = \frac{4}{3}$. Thus, $\delta_P(S) \leq \frac{3}{4}$ for $P \in A$. Moreover, for $P \in A \backslash B_1$ we have $h(v) \leq \frac{21v^2}{32}$ if $v \in [0,2]$. So $S(W^A_{\bullet,\bullet}; P) \leq \frac{7}{6} < \frac{4}{3}$. Thus, $\delta_P(S) = \frac{3}{4}$ if $P \in A \backslash B_1$. □

Lemma 7.1.26. *Suppose* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

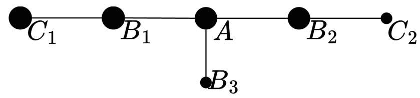


Figure 7.24: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{9}{13}$

Then $\tau(A) = 2$ *and the Zariski Decomposition of the divisor* $-K_S - vA$ *is given by:*

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B_2 - \frac{v}{3}(2B_1 + C_1) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B_2 - \frac{v}{3}(2B_1 + C_1) - (v-1)B_3 \text{ if } v \in [1,2], \\ -K_S - vA - \frac{v}{3}(2B_1 + C_1) - (v-1)(B_2 + B_3) - (v-2)C_2 \text{ if } v \in [2,3]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}B_2 + \frac{v}{3}(2B_1 + C_1) \text{ if } v \in [0,1], \\ \frac{v}{2}B_2 + \frac{v}{3}(2B_1 + C_1) + (v-1)B_3 \text{ if } v \in [1,2], \\ \frac{v}{3}(2B_1 + C_1) + (v-1)(B_2 + B_3) + (v-2)C_2 \text{ if } v \in [2,3]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3 - \frac{5v^2}{6} \text{ if } v \in [0,1], \\ \frac{v^2}{6} - 2v + 4 \text{ if } v \in [1,2], \\ \frac{2(3-v)^2}{3} \text{ if } v \in [2,3]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{5v}{6} \text{ if } v \in [0,1], \\ 1 - \frac{v}{6} \text{ if } v \in [1,2], \\ 2(1 - \frac{v}{3}) \text{ if } v \in [2,3]. \end{cases}$$

In this case $\delta_P(S) = \frac{9}{13}$ *if* $P \in A$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (3-v)A + 2B_1 + C_1 + 2B_2 + C_2 + 2B_3$. We have $S_S(A) = \frac{13}{9}$. Thus, $\delta_P(S) \leq \frac{9}{13}$ for $P \in A$. Moreover, for $P \in A$ we have if $P \in A \backslash (B_1 \cup B_2)$ or if $P \in A \backslash (B_1 \cup B_3)$ or if $P \in A \backslash (B_2 \cup B_3)$:

$$h(v) = \begin{cases} \frac{25v^2}{72} \text{ if } v \in [0,1], \\ \frac{(6-v)(11v-6)}{72} \text{ if } v \in [1,2], \\ \frac{4v(3-v)}{9} \text{ if } v \in [2,3]. \end{cases} \quad \text{or } h(v) = \begin{cases} \frac{55v^2}{72} \text{ if } v \in [0,1], \\ \frac{(6-v)(5v+6)}{72} \text{ if } v \in [1,2], \\ \frac{4v(3-v)}{9} \text{ if } v \in [2,3]. \end{cases}$$

$$\text{or } h(v) = \begin{cases} \frac{65v^2}{72} \text{ if } v \in [0,1], \\ \frac{(6-v)(7v+6)}{72} \text{ if } v \in [1,2], \\ \frac{2(3-v)(v+3)}{9} \text{ if } v \in [2,3]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{23}{27} < \frac{13}{9}$ or $S(W^A_{\bullet,\bullet};P) \leq \frac{29}{27} < \frac{13}{9}$ or $S(W^A_{\bullet,\bullet};P) \leq \frac{23}{18} < \frac{13}{9}$. Thus, $\delta_P(S) = \frac{9}{13}$ if $P \in A$. □

Lemma 7.1.27. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

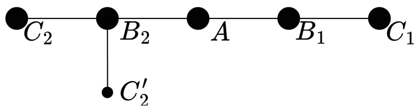


Figure 7.25: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{2}{3}$ with $-K_S - vA$ nef on $\left[0, \frac{3}{2}\right]$

Then $\tau(A) = 3$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2B_1 + C_1 + C_2 + 2B_2) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ -K_S - vA - \frac{v}{3}(2B_1 + C_1) - (v-1)(C_2 + 2B_2) - (2v-3)C_2' \text{ if } v \in \left[\frac{3}{2}, 3\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2B_1 + C_1 + C_2 + 2B_2) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{v}{3}(2B_1 + C_1) + (v-1)(C_2 + 2B_2) + (2v-3)C_2' \text{ if } v \in \left[\frac{3}{2}, 3\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3 - \frac{2v^2}{3} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{2(3-v)^2}{3} \text{ if } v \in \left[\frac{3}{2}, 3\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{2v}{3} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ 2(1 - \frac{v}{3}) \text{ if } v \in \left[\frac{3}{2}, 3\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{2}{3}$ if $P \in A \backslash B_2$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (3-v)A + 2B_1 + C_1 + 4B_2 + 2C_2 + 3C_2'$. We have $S_S(A) = \frac{3}{2}$. Thus, $\delta_P(S) \leq \frac{2}{3}$ for $P \in A$. Moreover, for $P \in A \backslash B_2$ we have:

$$h(v) \leq \begin{cases} \frac{2v^2}{3} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{2(3-v)(v+3)}{9} \text{ if } v \in \left[\frac{3}{2}, 3\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{4}{3} < \frac{3}{2}$. Thus, $\delta_P(S) = \frac{2}{3}$ if $P \in A \backslash B_2$. □

Lemma 7.1.28. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

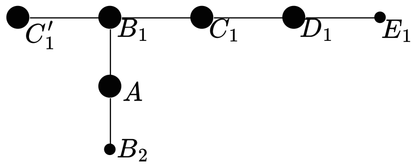


Figure 7.26: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{2}{3}$ with $-K_S - vA$ nef on $[0,1]$

Then $\tau(A) = 3$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{5}(6B_1 + 4C_1 + 2D_1 + 3C_1') \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{5}(6B_1 + 4C_1 + 2D_1 + 3C_1') - (v-1)B_2 \text{ if } v \in \left[1, \frac{5}{2}\right], \\ -K_S - vA - (v-1)(C_1' + B_2) - (2v-2)B_1 - (2v-3)C_1 - (2v-4)D_1 - (2v-5)E_1 \text{ if } v \in \left[\frac{5}{2}, 3\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(6B_1 + 4C_1 + 2D_1 + 3C_1') \text{ if } v \in [0,1], \\ \frac{v}{5}(6B_1 + 4C_1 + 2D_1 + 3C_1') + (v-1)B_2 \text{ if } v \in [1, \frac{5}{2}], \\ (v-1)(C_1' + B_2) + (2v-2)B_1 + (2v-3)C_1 + (2v-4)D_1 + (2v-5)E_1 \text{ if } v \in \left[\frac{5}{2}, 3\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3 - \frac{4v^2}{5} \text{ if } v \in [0,1], \\ \frac{v^2}{5} - 2v + 4 \text{ if } v \in \left[1, \frac{5}{2}\right], \\ (3-v)^2 \text{ if } v \in \left[\frac{5}{2}, 3\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{4v}{5} \text{ if } v \in [0,1], \\ 1 - \frac{v}{5} \text{ if } v \in \left[1, \frac{5}{2}\right], \\ 3 - v \text{ if } v \in \left[\frac{5}{2}, 3\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{2}{3}$ if $P \in A \backslash B_1$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (3-v)A + 2C_1' + 4B_1 + 3C_1 + 2D_1 + E_1 + 2B_2$. We have $S_S(A) = \frac{3}{2}$. Thus, $\delta_P(S) \leq \frac{2}{3}$ for $P \in A$. Moreover, for $P \in A \backslash B_1$ we have:

$$h(v) \leq \begin{cases} \frac{16v^2}{50} \text{ if } v \in [0,1], \\ \frac{(5-v)(9v-5)}{50} \text{ if } v \in \left[1, \frac{5}{2}\right], \\ \frac{(3-v)(v+1)}{2} \text{ if } v \in \left[\frac{5}{2}, 3\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \leq \frac{8}{9} < \frac{3}{2}$. Thus, $\delta_P(S) = \frac{2}{3}$ if $P \in A \backslash B_1$. □

Lemma 7.1.29. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

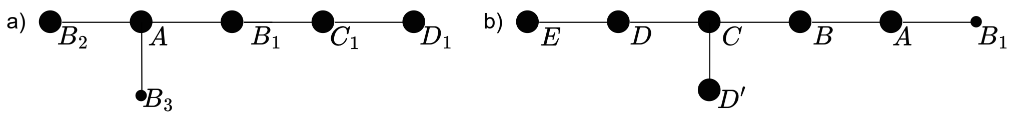


Figure 7.27: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{3}{5}$ with $\tau(A) = 4$

Then $\tau(A)=4$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

a).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{4}(2B_2+3B_1+2C_1+D_1) \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{4}(2B_2+3B_1+2C_1+D_1)-(v-1)B_3 \text{ if } v\in[1,4].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{4}(2B_2+3B_1+2C_1+D_1) \text{ if } v\in[0,1],\\ \frac{v}{4}(2B_2+3B_1+2C_1+D_1)+(v-1)B_3 \text{ if } v\in[1,4].\end{cases}$$

b).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{4}(5B+6C+4D+2E+3D') \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{4}(5B+6C+4D+2E+3D')-(v-1)B_1 \text{ if } v\in[1,4].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{4}(5B+6C+4D+2E+3D') \text{ if } v\in[0,1],\\ \frac{v}{4}(5B+6C+4D+2E+3D')+(v-1)B_1 \text{ if } v\in[1,4].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}\frac{3(2-v)(2+v)}{4} \text{ if } v\in[0,1],\\ \frac{(4-v)^2}{4} \text{ if } v\in[1,4].\end{cases} \quad P(v)\cdot A=\begin{cases}\frac{3v}{4} \text{ if } v\in[0,1],\\ 1-\frac{v}{4} \text{ if } v\in[1,4].\end{cases}$$

In this case $\delta_P(S)=\frac{3}{5}$ if $P\in A\backslash B$.

Proof. The Zariski Decomposition in part a). follows from $-K_S-vA\sim_{\mathbb{R}}(4-v)A+2B_2+3B_1+2C_1+D_1+3B_3$. A similar statement holds in other parts. We have $S_S(A)=\frac{5}{3}$. Thus, $\delta_P(S)\leq\frac{3}{5}$ for $P\in A$. Moreover, for $P\in A$ we have:

$$h(v)\leq\begin{cases}\frac{27v^2}{32} \text{ if } v\in[0,1],\\ \frac{(4-v)(5v+4)}{32} \text{ if } v\in[1,4].\end{cases} \quad \text{or } h(v)\leq\begin{cases}\frac{9v^2}{32} \text{ if } v\in[0,1],\\ \frac{(4-v)(7v-4)}{32} \text{ if } v\in[1,4].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\leq\frac{3}{2}<\frac{5}{3}$ or $S(W^A_{\bullet,\bullet};P)\leq 1<\frac{5}{3}$. Thus, $\delta_P(S)=\frac{3}{5}$ if $P\in A$. □

Lemma 7.1.30. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

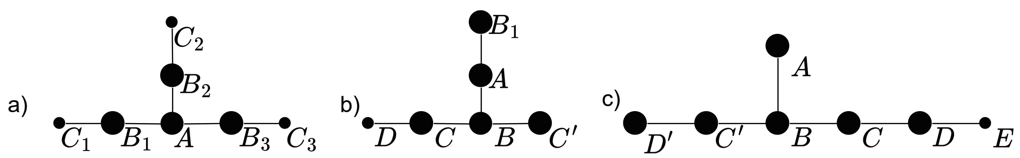


Figure 7.28: Dual graph: $(-K_S)^2=3$ and $\delta_P(S)=\frac{3}{5}$ with $\tau(A)=3$

Then $\tau(A)=3$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

a).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}(B_1+B_2+B_3) \text{ if } v\in[0,2],\\ -K_S-vA-(v-1)(B_1+B_2+B_3)-(v-2)(C_1+C_2+C_3) \text{ if } v\in[2,3].\end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(B_1+B_2+B_3) \textit{ if } v \in [0,2], \\ (v-1)(B_1+B_2+B_3)+(v-2)(C_1+C_2+C_3) \textit{ if } v \in [2,3]. \end{cases}$$

b). $$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}(B_1+2B+C+C') \textit{ if } v \in [0,2], \\ -K_S - vA - (v-1)(B_1+C') - (v-2)C_1 - (2v-2)B - (2v-3)C - (2v-4)D \textit{ if } v \in [2,3]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(B_1+2B+C+C') \textit{ if } v \in [0,2], \\ (v-1)(B_1+C') + (v-2)C_1 + (2v-2)B + (2v-3)C + (2v-4)D \textit{ if } v \in [2,3]. \end{cases}$$

c). $$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}(D'+2C'+3B+2C+D) \textit{ if } v \in [0,2], \\ -K_S - vA - (v-1)(D'+2C'+B) - (3v-4)C - (3v-5)D - (3v-6)E \textit{ if } v \in [2,3]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(D'+2C'+3B+2C+D) \textit{ if } v \in [0,2], \\ (v-1)(D'+2C'+B) + (3v-4)C + (3v-5)D + (3v-6)E \textit{ if } v \in [2,3]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3 - \frac{v^2}{2} \textit{ if } v \in [0,2], \\ (3-v)^2 \textit{ if } v \in [2,3]. \end{cases} \qquad P(v)\cdot A = \begin{cases} \frac{v}{2} \textit{ if } v \in [0,2], \\ 3-v \textit{ if } v \in [2,3]. \end{cases}$$

In this case $\delta_P(S) = \frac{3}{5}$ *if* $P \in A \backslash B$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} (3-v)A + 2B_1 + C_1 + 2B_2 + C_2 + 2B_3 + C_3$. A similar statement holds in other parts. We have $S_S(A) = \frac{5}{3}$. Thus, $\delta_P(S) \leq \frac{3}{5}$ for $P \in A$. Moreover, for $P \in A \backslash B$ we have:

$$h(v) \leq \begin{cases} \frac{3v^2}{8} \text{ if } v \in [0,2], \\ \frac{(3-v)(v+1)}{2} \text{ if } v \in [2,3]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{11}{9} < \frac{5}{3}$. Thus, $\delta_P(S) = \frac{3}{5}$ if $P \in A \backslash B$. □

Lemma 7.1.31. *Suppose* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

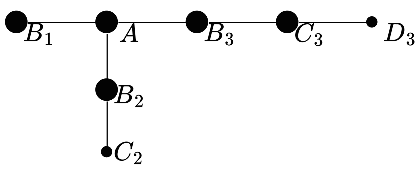


Figure 7.29: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{9}{19}$

Then $\tau(A)=4$ *and the Zariski Decomposition of the divisor* $-K_S-vA$ *is given by:*

$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}(B_1+B_2)-\frac{v}{3}(2B_3+C_3) \text{ if } v\in[0,2],\\ -K_S-vA-\frac{v}{2}B_1-\frac{v}{3}(2B_3+C_3)-(v-1)B_2-(v-2)C_2 \text{ if } v\in[2,3],\\ -K_S-vA-\frac{v}{2}B_1-(v-1)(B_2+B_3)-(v-2)(C_2+C_3)-(v-3)D_3 \text{ if } v\in[3,4].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{2}(B_1+B_2)+\frac{v}{3}(2B_3+C_3) \text{ if } v\in[0,2],\\ \frac{v}{2}B_1+\frac{v}{3}(2B_3+C_3)+(v-1)B_2+(v-2)C_2 \text{ if } v\in[2,3],\\ \frac{v}{2}B_1+(v-1)(B_2+B_3)+(v-2)(C_2+C_3)+(v-3)D_3 \text{ if } v\in[3,4].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}\frac{(3-v)(3+v)}{3} \text{ if } v\in[0,2],\\ \frac{v^2}{6}-2v+5 \text{ if } v\in[2,3],\\ \frac{(4-v)^2}{2} \text{ if } v\in[3,4].\end{cases}\quad P(v)\cdot A=\begin{cases}\frac{v}{3} \text{ if } v\in[0,2],\\ 1-\frac{v}{6} \text{ if } v\in[2,3],\\ 2-\frac{v}{2} \text{ if } v\in[3,4].\end{cases}$$

In this case $\delta_P(S)=\frac{9}{19}$ *if* $P\in A$.

Proof. The Zariski Decomposition follows from $-K_S-vA\sim_{\mathbb{R}}(4-v)A+2B_1+3B_2+2C_2+3B_3+2C_3+D_3$. We have $S_S(A)=\frac{19}{9}$. Thus, $\delta_P(S)\leq\frac{9}{19}$ for $P\in A$. Moreover, if $P\in A\backslash(B_1\cup B_2)$ or if $P\in A\cap B_2$ or if $P\in A\cap B_1$ we have:

$$h(v)=\begin{cases}\frac{5v^2}{18} \text{ if } v\in[0,2],\\ \frac{(6-v)(7v+6)}{72} \text{ if } v\in[2,3],\\ \frac{3(4-v)v}{8} \text{ if } v\in[3,4].\end{cases}\quad \text{or } h(v)\leq\begin{cases}\frac{2v^2}{9} \text{ if } v\in[0,2],\\ \frac{(6-v)(11v-6)}{72} \text{ if } v\in[2,3],\\ \frac{3(4-v)v}{8} \text{ if } v\in[3,4].\end{cases}$$

$$\text{or } h(v)\leq\begin{cases}\frac{2v^2}{9} \text{ if } v\in[0,2],\\ \frac{(6-v)(5v+6)}{72} \text{ if } v\in[2,3],\\ \frac{(4-v)(v+4)}{8} \text{ if } v\in[3,4].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\leq\frac{5}{3}<\frac{19}{9}$ or $S(W^A_{\bullet,\bullet};P)\leq\frac{3}{2}<\frac{19}{9}$ or $S(W^A_{\bullet,\bullet};P)\leq\frac{35}{27}<\frac{19}{9}$. Thus, $\delta_P(S)=\frac{19}{9}$ if $P\in A$. □

Lemma 7.1.32. *Suppose* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

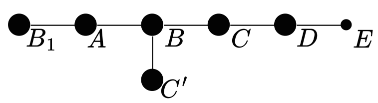


Figure 7.30: Dual graph: $(-K_S)^2=3$ and $\delta_P(S)=\frac{6}{13}$

Then $\tau(A) = 4$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{5}(6B+4C+2D+3C') - \frac{v}{2}B_1 \text{ if } v \in \left[0, \frac{5}{2}\right], \\ -K_S - vA - (v-1)C' - (2v-2)B - (2v-3)C - (2v-4)D - (2v-5)E - \frac{v}{2}B_1 \text{ if } v \in \left[\frac{5}{2}, 4\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(6B+4C+2D+3C') + \frac{v}{2}B_1 \text{ if } v \in \left[0, \frac{5}{2}\right], \\ (v-1)C' + (2v-2)B + (2v-3)C + (2v-4)D + (2v-5)E + \frac{v}{2}B_1 \text{ if } v \in \left[\frac{5}{2}, 4\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3 - \frac{3v^2}{10} \text{ if } v \in \left[0, \frac{5}{2}\right], \\ \frac{(4-v)^2}{2} \text{ if } v \in \left[\frac{5}{2}, 4\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{3v}{10} \text{ if } v \in \left[0, \frac{5}{2}\right], \\ 2 - \frac{v}{2} \text{ if } v \in \left[\frac{5}{2}, 4\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{6}{13}$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (4-v)A + 6B + 5C + 4D + 3E + 3C' + 2B_1$. We have $S_S(A) = \frac{13}{6}$. Thus, $\delta_P(S) \leq \frac{6}{13}$ for $P \in A$. Moreover, for $P \in A \backslash B$ we have:

$$h(v) \leq \begin{cases} \frac{39v^2}{200} \text{ if } v \in \left[0, \frac{5}{2}\right], \\ \frac{(4-v)(v+4)}{8} \text{ if } v \in \left[\frac{5}{2}, 4\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \leq \frac{4}{3} < \frac{13}{6}$. Thus, $\delta_P(S) = \frac{6}{13}$ if $P \in A \backslash B$. □

Lemma 7.1.33. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

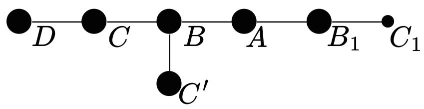


Figure 7.31: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{3}{7}$

Then $\tau(A) = 5$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{5}(6B+4C+2D+3C') - \frac{v}{2}B_1 \text{ if } v \in [0,2], \\ -K_S - vA - \frac{v}{5}(6B+4C+2D+3C') - (v-1)B_1 - (v-2)C_1 \text{ if } v \in [2,5]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(6B+4C+2D+3C') + \frac{v}{2}B_1 \text{ if } v \in [0,2], \\ \frac{v}{5}(6B+4C+2D+3C') + (v-1)B_1 + (v-2)C_1 \text{ if } v \in [2,5]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3 - \frac{3v^2}{10} \text{ if } v \in [0,2], \\ \frac{(5-v)^2}{5} \text{ if } v \in [2,5]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{3v}{10} \text{ if } v \in [0,2], \\ 1 - \frac{v}{5} \text{ if } v \in [2,5]. \end{cases}$$

In this case $\delta_P(S) = \frac{3}{7}$ *if* $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (5-v)A + 6B + 4C + 2D + 3C' + 4B_1 + 3C_1$. We have $S_S(A) = \frac{7}{3}$. Thus, $\delta_P(S) \leq \frac{3}{7}$ for $P \in A$. Moreover, for $P \in A \backslash B$ we have:

$$h(v) \leq \begin{cases} \frac{39v^2}{200} \text{ if } v \in [0,2], \\ \frac{(5-v)(9v-5)}{50} \text{ if } v \in [2,5]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \leq \frac{5}{3} < \frac{7}{3}$. Thus, $\delta_P(S) = \frac{3}{7}$ if $P \in A \backslash B$. □

Lemma 7.1.34. *Suppose* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

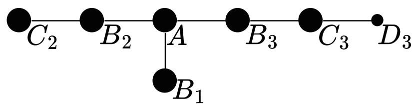


Figure 7.32: Dual graph: $(-K_S)^2 = 3$ and $\delta_P(S) = \frac{1}{3}$

Then $\tau(A) = 6$ *and the Zariski Decomposition of the divisor* $-K_S - vA$ *is given by:*

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B_1 - \frac{v}{3}(2B_2 + C_2 + 2B_3 + C_3) \text{ if } v \in [0,3], \\ -K_S - vA - \frac{v}{2}B_1 - \frac{v}{3}(2B_2 + C_2) - (v-1)B_3 - (v-2)C_3 - (v-3)D_3 \text{ if } v \in [3,6]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}B_1 + \frac{v}{3}(2B_2 + C_2 + 2B_3 + C_3) \text{ if } v \in [0,3], \\ \frac{v}{2}B_1 + \frac{v}{3}(2B_2 + C_2) + (v-1)B_3 + (v-2)C_3 + (v-3)D_3 \text{ if } v \in [3,6]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 3 - \frac{v^2}{6} \text{ if } v \in [0,3], \\ \frac{(6-v)^2}{6} \text{ if } v \in [3,6]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{v}{6} \text{ if } v \in [0,3], \\ 1 - \frac{v}{6} \text{ if } v \in [3,6]. \end{cases}$$

In this case $\delta_P(S) = \frac{1}{3}$ *if* $P \in A$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (6-v)A + 3B_1 + 4B_2 + 2C_2 + 5B_3 + 4C_3 + 3D_3$. We have $S_S(A) = 3$. Thus, $\delta_P(S) \leq \frac{1}{3}$ for $P \in A$. Moreover, if $P \in A \backslash (B_1 \cup B_3)$ or if $P \in A \cap B_3$ or if $P \in A \cap B_1$ we have:

$$h(v) \leq \begin{cases} \frac{v^2}{8} \text{ if } v \in [0,3], \\ \frac{(6-v)(7v+6)}{72} \text{ if } v \in [3,6]. \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{v^2}{8} \text{ if } v \in [0,3], \\ \frac{(6-v)(11v-6)}{72} \text{ if } v \in [3,6]. \end{cases}$$

$$\text{or } h(v) \leq \begin{cases} \frac{7v^2}{72} \text{ if } v \in [0,3], \\ \frac{(6-v)(5v+6)}{72} \text{ if } v \in [3,6]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{13}{6} < 3$ or $S(W^A_{\bullet,\bullet};P) \leq \frac{7}{3} < 3$ or $S(W^A_{\bullet,\bullet};P) \leq \frac{5}{3} < 3$. Thus, $\delta_P(S) = \frac{1}{3}$ if $P \in A$. □

7.2 Finding δ-invariants for degree 3

Let X be a singular del Pezzo surface of degree 3 with and S be a minimal resolution of X. Then there are several possible cases:

I. X has an $\mathbb{A}_1$ singularity and contains 21 lines. In this case, we let E be the exceptional divisor, L_i for $i \in \{1,2,3,4,5,6\}$ be the lines on S,

II. X has two $\mathbb{A}_1$ singularities and contains 16 lines. In this case, we let E_1 and E_2 be the exceptional divisors, $L_{i,j}$ for $i \in \{1,2\}$, $j \in \{1,2,3,4\}$ be the lines on S,

III. X has three $\mathbb{A}_1$ singularities and contains 12 lines. In this case, we let E_1, E_2 and E_3 be the exceptional divisors, L_{12}, L_{23}, L_{13}, $L_{i,j}$ for $i \in \{1,2,3\}$, $j \in \{1,2\}$ be the lines on S,

IV. X has four $\mathbb{A}_1$ singularities and contains 9 lines. In this case, we let E_1, E_2, E_3 and E_4 be the exceptional divisors, L_{ij} for $(i,j) \in \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$ be the lines on S,

V. X has $\mathbb{A}_2$ singularity and contains 15 lines. In this case, we let E_1 and E_1' be the exceptional divisors, $L_{1,i}$ and $L_{1,i}'$ for $i \in \{1,2,3\}$ be the lines on S,

VI. X has $\mathbb{A}_2$ and $\mathbb{A}_1$ singularities and contains 11 lines. In this case, we let E_1, E_1' and E_2 be the exceptional divisors, L_{12}, L_1, $L_{i,j}$ for $i \in \{1,2\}$ and $j \in \{1,2,3\}$ be the lines on S,

VII. X has $\mathbb{A}_2$ and two $\mathbb{A}_1$ singularities and contains 8 lines. In this case, we let E_1, E_1', E_2 and E_2' be the exceptional divisors, L_{12}, L_{12}', $L_{i,1}$ and $L_{i,1}'$ for $i \in \{1,2\}$ be the lines on S

VIII. X has two $\mathbb{A}_2$ singularities and contains 7 lines. In this case, we let E_1, E_1', E_2 and E_2' be the exceptional divisors, L_{12}, $L_{i,j}$ for $i \in \{1,2\}$ and $j \in \{1,2,3\}$ be the lines on S,

IX. X has two $\mathbb{A}_2$ and $\mathbb{A}_1$ singularities and contains 5 lines. In this case, we let E_1, E_1', E_2, E_2' and E_3 be the exceptional divisors, L_{12}, L_{13}, L_{23}, $L_{1,1}$ and $L_{2,1}$ be the lines on S,

X. X has three $\mathbb{A}_2$ singularities and contains 3 lines. In this case, we let E_1, E_1', E_2, E_2', E_3 and E_3' be the exceptional divisors, L_{12}, L_{13}, L_{23} be the lines on S,

XI. X has $\mathbb{A}_3$ singularity and contains 10 lines. In this case, we let E_1, E_2 and E_1' be the exceptional divisors, $L_{2,1}$, $L_{1,i}$ and $L_{1,i}'$ for $i \in \{1,2\}$ be the lines on S,

XII. X has $\mathbb{A}_3$ and $\mathbb{A}_1$ singularities and contains 7 lines. In this case, we let E_1, E_1', E_2 and E_3 be the exceptional divisors, L_{13}, $L_{2,1}$, $L_{i,1}$ and $L_{i,1}'$ for $i \in \{1,3\}$ be the lines on S,

XIII. X has $\mathbb{A}_3$ and two $\mathbb{A}_1$ singularities and contains 5 lines. In this case, we let E_1, E_1', E_2, E_3 and E_3' be the exceptional divisors, L_{13}, L_{13}', $L_{2,1}$ and L_3 be the lines on S,

XIV. X has $\mathbb{A}_4$ singularity and contains 6 lines. In this case, we let E_1, E_2, E_3 and E_4 be the exceptional divisors, $L_{1,1}$, $L_{3,1}$, $L_{4,1}$ and $L_{4,2}$ be the lines on S,

XV. X has $\mathbb{A}_4$ and $\mathbb{A}_1$ singularities and contains 4 lines. In this case, we let E_1, E_2, E_3, E_4 and E_5 be the exceptional divisors, $L_{1,1}$, $L_{3,1}$ and $L_{5,1}$ be the lines on S,

XVI. X has $\mathbb{A}_5$ singularity and contains 3 lines. In this case, we let E_1, E_2, E_3, E_4 and E_5 be the exceptional divisors, $L_{2,1}$, $L_{5,1}$ and $L_{5,2}$ be the lines on S,

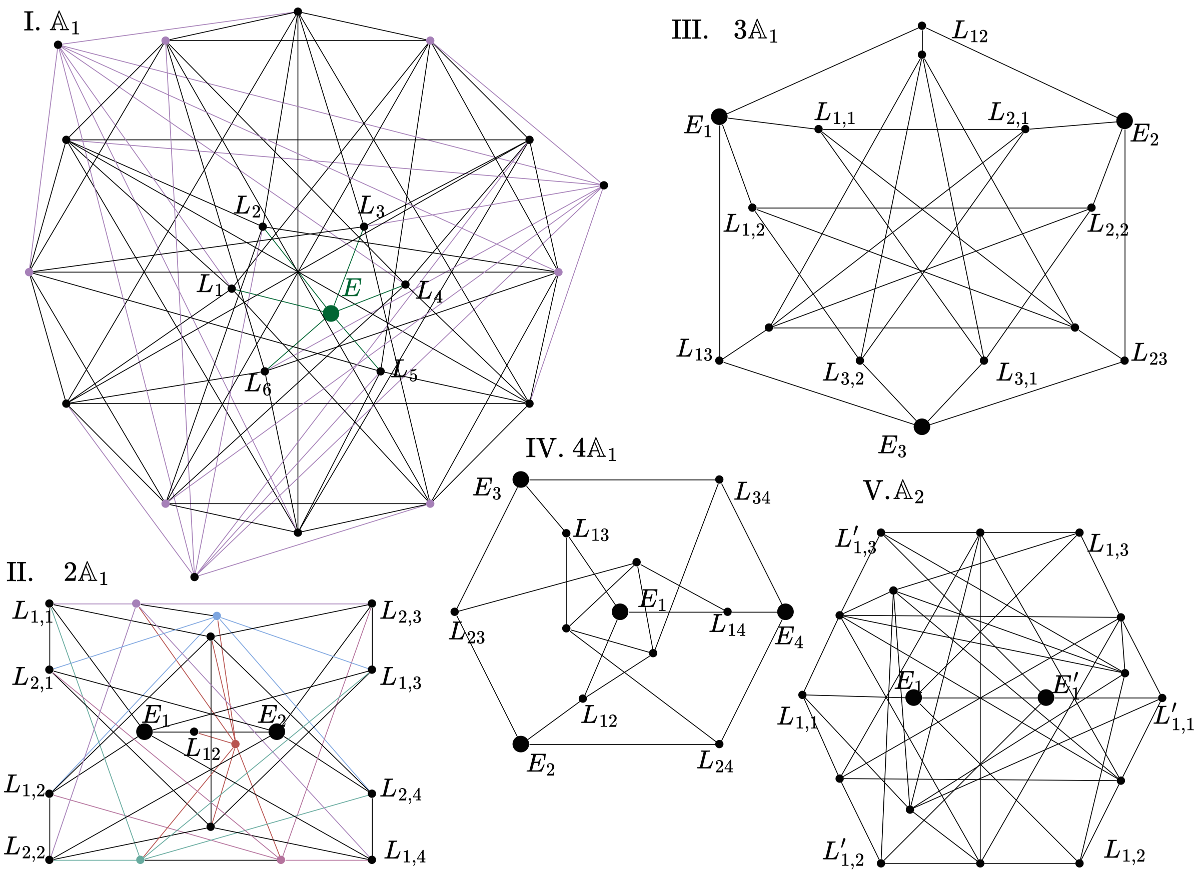
XVII. X has $\mathbb{A}_5$ and $\mathbb{A}_1$ singularities and contains 2 lines. In this case, we let E_1, E_2, E_3, E_4 and E_5 be the exceptional divisors, $L_{2,1}$, L_{56} be the lines on S,

XVIII. X has $\mathbb{D}_4$ singularity and contains 6 lines. In this case, we let E_1, E_2, E_3 and E be the exceptional divisors, $L_{i,1}$ for $i \in \{1,2,3\}$ be the lines on S,

XIX. X has $\mathbb{D}_5$ singularity and contains 3 lines. In this case, we let E_1, E_2, E_3, E_4 and E be the exceptional divisors, L and $L_{4,1}$ be the lines on S,

XX. X has $\mathbb{E}_6$ singularity and contains 1 line. In this case, we let E_1, E_2, E_3, E_4, E_5 and E be the exceptional divisors, $L_{5,1}$ be the line on S.

such that the dual graph of the (-1)-curves and (-2)-curves on S is given the picture below.



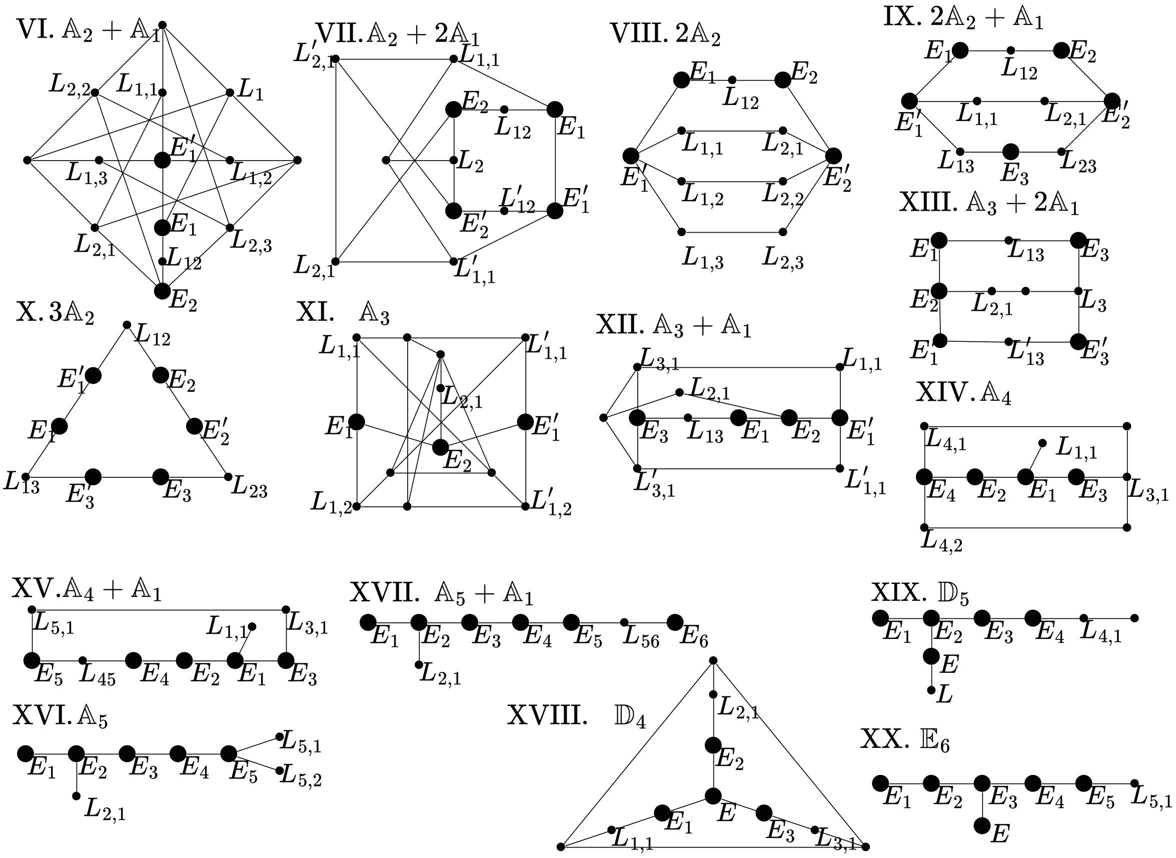


Figure 7.33: Du Val del Pezzo surfaces with $(-K_S)^2=3$

One has

I. $\delta(X)=\frac{6}{5}$ since depending on the position of point $P\in S$ we have

P	E	$\left(\bigcup_{i\in\{1\dots6\}} L_i\right)\backslash E$	o/w
$\delta_P(S)$	$\frac{6}{5}$	$\frac{27}{17}$	$\geq\frac{3}{2}$

Table 7.4: Local δ-invariants: $(-K_S)^2=3$ and $\mathbb{A}_1$ singularity

II. $\delta(X)=\frac{6}{5}$ since depending on the position of point $P\in S$ we have

P	$\mathbf{E}_2$	$L_{12}\backslash\mathbf{E}_2$	$\left(\bigcup_{i\in\{1,2\},j\in\{1,2,3,4\}} L_{i,j}\right)\backslash\Big(\mathbf{L}_2\cup\mathbf{E}_2\Big)$	$\mathbf{L}_2$	o/w
$\delta_P(S)$	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{27}{17}$	$\geq\frac{54}{35}$	$\geq\frac{3}{2}$

where $\mathbf{E}_2:=E_1\cup E_2$, $\mathbf{L}_2:=\bigcup_{k\in\{1,2,3,4\}}\left(L_{1,k}\cap L_{2,k}\right)$.

Table 7.5: Local δ-invariants: $(-K_S)^2=3$ and $2\mathbb{A}_1$ singularities

III. $\delta(X)=\frac{6}{5}$ since depending on the position of point $P\in S$ we have

P	$\mathbf{E}_3$	$(L_{12}\cup L_{23}\cup L_{13})\backslash\mathbf{E}_3$	$\left(\bigcup_{i\in\{1,2,3\},j\in\{1,2\}} L_{i,j}\right)\backslash\left(\mathbf{L}_3\cup\mathbf{E}_3\right)$	$\mathbf{L}_3$	o/w
$\delta_P(S)$	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{27}{17}$	$\geq\frac{54}{35}$	$\geq\frac{3}{2}$

where $\mathbf{E}_3 := E_1\cup E_2\cup E_3$, $\mathbf{L}_3 := \bigcup_{k\in\{1,2\},i_1,i_2\in\{1,2,3\},i_1\neq i_2}\left(L_{i_1,k}\cap L_{i_2,k}\right)$.

Table 7.6: Local δ-invariants: $(-K_S)^2=3$ and $3\mathbb{A}_1$ singularities

IV. $\delta(X)=\frac{6}{5}$ since depending on the position of point $P\in S$ we have

P	$\mathbf{E}_4$	$(L_{12}\cup L_{13}\cup L_{14}\cup L_{23}\cup L_{24}\cup L_{34})\backslash\mathbf{E}_4$	o/w
$\delta_P(S)$	$\frac{6}{5}$	$\frac{9}{7}$	$\geq\frac{3}{2}$

where $\mathbf{E}_4 := E_1\cup E_2\cup E_3\cup E_4$.

Table 7.7: Local δ-invariants: $(-K_S)^2=3$ and $4\mathbb{A}_1$ singularities

V. $\delta(X)=1$ since depending on the position of point $P\in S$ we have

P	$E_1\cup E_1'$	$\bigcup_{i\in\{1,2,3\}}\left(L_{1,i}\cup L_{1,i}'\right)\backslash(E_1\cup E_1')$	o/w
$\delta_P(S)$	1	$\frac{3}{2}$	$\geq\frac{3}{2}$

Table 7.8: Local δ-invariants: $(-K_S)^2=3$ and $\mathbb{A}_2$ singularity

VI. $\delta(X)=1$ since depending on the position of point $P\in S$ we have

P	$\mathbf{E}_6^{(1)}$	$(E_2\cup L_{12})\backslash E_1$	$\left(\bigcup_{i\in\{1,2,3\}} L_{1,i}\cup L_1\right)\backslash\mathbf{E}_6^{(1)}$	$(L_{2,1}\cup L_{2,1}')\backslash\mathbf{E}_6^{(2)}$	o/w
$\delta_P(S)$	1	$\frac{6}{5}$	$\frac{3}{2}$	$\frac{27}{17}$	$\geq\frac{3}{2}$

where $\mathbf{E}_6^{(1)} := E_1\cup E_1'$, $\mathbf{E}_6^{(2)} := E_2\cup E_2'$.

Table 7.9: Local δ-invariants: $(-K_S)^2=3$ and $\mathbb{A}_2\mathbb{A}_1$ singularities

VII. $\delta(X)=1$ since depending on the position of point $P\in S$ we have

P	$\mathbf{E}_7^{(1)}$	$(\mathbf{E}_7^{(2)}\cup L_{12}\cup L_{12}')\backslash\mathbf{E}_7^{(1)}$	$L_2\backslash\mathbf{E}_7^{(2)}$	$\mathbf{L}_7^{(1)}\backslash\mathbf{E}_7^{(1)}$	$\mathbf{L}_7^{(2)}\backslash\mathbf{E}_7^{(2)}$	o/w
$\delta_P(S)$	1	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{3}{2}$	$\frac{27}{17}$	$\geq\frac{3}{2}$

where $\mathbf{E}_7^{(1)} := E_1\cup E_1'$, $\mathbf{E}_7^{(2)} := E_2\cup E_2'$, $\mathbf{L}_7^{(1)} := L_{1,1}\cup L_{1,1}'$, $\mathbf{L}_7^{(2)} := L_{2,1}\cup L_{2,1}'$.

Table 7.10: Local δ-invariants: $(-K_S)^2=3$ and $\mathbb{A}_2 2\mathbb{A}_1$ singularities

VIII. $\delta(X)=1$ since depending on the position of point $P\in S$ we have

P	$E_1\cup E_1'\cup E_2\cup E_2'\cup L_{12}$	$\left(\bigcup_{i\in\{1,2\},j\in\{1,2,3\}} L_{i,j}\right)\backslash(E_1'\cup E_2')$	o/w
$\delta_P(S)$	1	$\frac{3}{2}$	$\geq\frac{3}{2}$

Table 7.11: Local δ-invariants: $(-K_S)^2=3$ and $2\mathbb{A}_2$ singularities

IX. $\delta(X)=1$ since depending on the position of point $P\in S$ we have

P	$\mathbf{E}_9\cup L_{12}$	$(E_3\cup L_{13}\cup L_{23})\backslash(E_1'\cup E_2')$	$(L_{1,1}\cup L_{2,1})\backslash(E_1'\cup E_2')$	o/w
$\delta_P(S)$	1	$\frac{6}{5}$	$\frac{3}{2}$	$\geq\frac{3}{2}$

where $\mathbf{E}_9:=E_1\cup E_1'\cup E_2\cup E_2'\cup L_{12}$.

Table 7.12: Local δ-invariants: $(-K_S)^2=3$ and $2\mathbb{A}_2\mathbb{A}_1$ singularities

X. $\delta(X)=1$ since depending on the position of point $P\in S$ we have

P	$E_1\cup E_1'\cup E_2\cup E_2'\cup E_3\cup E_3'\cup L_{12}\cup L_{13}\cup L_{23}$	o/w
$\delta_P(S)$	1	$\geq\frac{3}{2}$

Table 7.13: Local δ-invariants: $(-K_S)^2=3$ and $3\mathbb{A}_2$ singularities

XI. $\delta(X)=\frac{9}{11}$ since depending on the position of point $P\in S$ we have

P	E_2	$(E_1\cup E_1')\backslash E_2$	$L_{2,1}\backslash E_2$	$\bigcup_{i\in\{1,2\}}\left(L_{1,i}\cup L_{1,i}'\right)\backslash(E_1\cup E_1')$	o/w
$\delta_P(S)$	$\frac{9}{11}$	$\frac{18}{19}$	$\frac{9}{7}$	$\frac{54}{37}$	$\geq\frac{3}{2}$

Table 7.14: Local δ-invariants: $(-K_S)^2=3$ and $\mathbb{A}_3$ singularity

XII. $\delta(X)=\frac{9}{11}$ since depending on the position of point $P\in S$ we have

P	E_2	$\mathbf{E}_{12}\backslash E_2$	$L_{13}\backslash E_1$	$E_3\backslash L_{13}$	$L_{2,1}\backslash E_2$	$\mathbf{L}_{12}^{(1)}\backslash E_1'$	$\mathbf{L}_{12}^{(3)}\backslash(E_3\cup\mathbf{L}_{12}^{(1)})$	o/w
$\delta_P(S)$	$\frac{9}{11}$	$\frac{18}{19}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{54}{37}$	$\frac{27}{17}$	$\geq\frac{3}{2}$

where $\mathbf{E}_{12}:=E_1\cup E_1'$, $\mathbf{L}_{12}^{(1)}:=L_{1,1}\cup L_{1,1}'$, $\mathbf{L}_{12}^{(3)}:=L_{3,1}\cup L_{3,1}'$.

Table 7.15: Local δ-invariants: $(-K_S)^2=3$ and $\mathbb{A}_3\mathbb{A}_1$ singularities

XIII. $\delta(X)=\frac{9}{11}$ since depending on the position of point $P\in S$ we have

P	E_2	$\mathbf{E}_{13}^{(1)}\backslash E_2$	$\mathbf{L}_{13}\backslash\mathbf{E}_{13}^{(1)}$	$\mathbf{E}_{13}^{(3)}\backslash\mathbf{L}_{13}$	$(L_{2,1}\cup L_3)\backslash(E_2\cup\mathbf{E}_{13}^{(3)})$	o/w
$\delta_P(S)$	$\frac{9}{11}$	$\frac{18}{19}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{9}{7}$	$\geq\frac{3}{2}$

where $\mathbf{E}_{13}^{(1)}:=E_1\cup E_1'$, $\mathbf{E}_{13}^{(3)}:=E_3\cup E_3'$, $\mathbf{L}_{13}:=L_{13}\cup L_{13}'$.

Table 7.16: Local δ-invariants: $(-K_S)^2=3$ and $\mathbb{A}_3 2\mathbb{A}_1$ singularities

XIV. $\delta(X)=\frac{9}{13}$ since depending on the position of point $P\in S$ we have

P	E_1	$E_2\backslash E_1$	$E_3\backslash E_1$	$E_4\backslash E_2$	$L_{1,1}\backslash E_1$	$L_{3,1}\backslash E_3$	$(L_{4,1}\cup L_{4,2})\backslash E_4$	o/w
$\delta_P(S)$	$\frac{9}{13}$	$\frac{18}{23}$	$\frac{27}{29}$	$\frac{9}{10}$	$\frac{27}{23}$	$\frac{27}{19}$	$\frac{36}{25}$	$\geq\frac{3}{2}$

Table 7.17: Local δ-invariants: $(-K_S)^2=3$ and $\mathbb{A}_4$ singularity

XV. $\delta(X)=\frac{9}{13}$ since depending on the position of point $P\in S$ we have

P	E_1	$E_2\backslash E_1$	$E_3\backslash E_1$	$E_4\backslash E_2$	$L_{1,1}\backslash E_1$
$\delta_P(S)$	$\frac{9}{13}$	$\frac{18}{23}$	$\frac{27}{29}$	$\frac{9}{10}$	$\frac{27}{23}$

P	$L_{45}\backslash E_4$	$E_5\backslash L_{45}$	$L_{3,1}\backslash E_3$	$L_{5,1}\backslash(L_{3,1}\cup E_5)$	o/w
$\delta_P(S)$	$\frac{18}{17}$	$\frac{6}{5}$	$\frac{27}{19}$	$\frac{27}{17}$	$\geq\frac{3}{2}$

Table 7.18: Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{A}_4\mathbb{A}_1$ singularities

XVI. $\delta(X) = \frac{3}{5}$ since depending on the position of point $P \in S$ we have

P	E_2	$E_3\backslash E_2$	$E_4\backslash E_3$	$E_5\backslash E_4$	$E_1\backslash E_2$	$L_{2,1}\backslash E_2$	$(L_{5,1}\cup L_{5,2})\backslash E_5$	o/w
$\delta_P(S)$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{6}{7}$	$\frac{12}{13}$	1	$\frac{10}{7}$	$\geq\frac{3}{2}$

Table 7.19: Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{A}_5$ singularity

XVII. $\delta(X) = \frac{3}{5}$ since depending on the position of point $P \in S$ we have

P	E_2	$E_3\backslash E_2$	$E_4\backslash E_3$	$E_5\backslash E_4$	$E_1\backslash E_2$	$(L_{56}\cup L_{2,1})\backslash(E_2\cup E_5)$	$E_6\backslash L_{56}$	o/w
$\delta_P(S)$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{6}{7}$	$\frac{12}{13}$	1	$\frac{6}{5}$	$\geq\frac{3}{2}$

Table 7.20: Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{A}_5\mathbb{A}_1$ singularities

XVIII. $\delta(X) = \frac{3}{5}$ since depending on the position of point $P \in S$ we have

P	E	$(E_1\cup E_2\cup E_3)\backslash E$	$(L_{1,1}\cup L_{2,1}\cup L_{3,1})\backslash(E_1\cup E_2\cup E_3)$	otherwise
$\delta_P(S)$	$\frac{3}{5}$	$\frac{9}{11}$	$\frac{9}{7}$	$\geq\frac{3}{2}$

Table 7.21: Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{D}_4$ singularity

XIX. $\delta(X) = \frac{9}{19}$ since depending on the position of point $P \in S$ we have

P	E_2	$E_3\backslash E_2$	$E\backslash E_2$	$E_1\backslash E_2$	$E_4\backslash E_3$	$L\backslash E$	$L_{4,1}\backslash E_4$	o/w
$\delta_P(S)$	$\frac{9}{19}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{27}{35}$	$\frac{9}{11}$	$\frac{9}{8}$	$\frac{9}{7}$	$\geq\frac{3}{2}$

Table 7.22: Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{D}_5$ singularity

XX. $\delta(X) = \frac{1}{3}$ since depending on the position of point $P \in S$ we have

P	E_3	$E_4\backslash E_3$	$E_2\backslash E_3$	$(L_{123}\cup E_5)\backslash(E_3\cup E_4)$	$E_1\backslash E_2$	$E_6\backslash E_5$	o/w
$\delta_P(S)$	$\frac{1}{3}$	$\frac{3}{7}$	$\frac{6}{13}$	$\frac{3}{5}$	$\frac{3}{4}$	1	$\geq\frac{3}{2}$

Table 7.23: Local δ-invariants: $(-K_S)^2 = 3$ and $\mathbb{E}_6$ singularity

Proof. We prove each case separately using lemmas from the previous section.

I. If $P \in E$, the assertion follows from Lemma 7.1.10 [a).]. If $P \in \left(\bigcup_{i\in\{1,.6\}} L_i\right)\backslash E$, the assertion follows from Lemma 7.1.2 [a).]. If P is a general point, the assertion follows from Lemma 7.1.1.

II. If $P \in E_1 \cup E_2$, the assertion follows from Lemma 7.1.10 [b).]. If $P \in L_{12} \backslash (E_1 \cup E_2)$, the assertion follows from Lemma 7.1.9 [a).]. If $P \in \left(\bigcup_{i\in\{1,2\},j\in\{1,2,3,4\}} L_{i,j}\right) \backslash \Big(\bigcup_{j\in\{1,2,3,4\}} \left(L_{1,j} \cap L_{2,j}\right) \cup E_1 \cup E_2\Big)$, the assertion follows from Lemma 7.1.2 [b).]. If $P \in \bigcup_{j\in\{1,2,3,4\}} \left(L_{1,j} \cap L_{2,j}\right)$, the assertion follows from Remark 7.1.3. If P is a general point, the assertion follows from Lemma 7.1.1.

III. If $P \in E_1 \cup E_2 \cup E_3$, the assertion follows from Lemma 7.1.10 [c).]. If $P \in (L_{12} \cup L_{23} \cup L_{13}) \backslash (E_1 \cup E_2 \cup E_3)$, the assertion follows from Lemma 7.1.9 [a).]. If $P \in \left(\bigcup_{i\in\{1,2,3\},j\in\{1,2\}} L_{i,j}\right) \backslash \Big(\bigcup_{k\in\{1,2\}} \left(L_{1,k} \cap L_{2,k}\right) \cup E_1 \cup E_2 \cup E_3\Big)$, the assertion follows from Lemma 7.1.2 [c).]. If $P \in \bigcup_{k\in\{1,2\}} \left(L_{1,k} \cap L_{2,k}\right)$. In this case we apply Remark 7.1.3. If P is a general point, the assertion follows from Lemma 7.1.1.

IV. If $P \in E_1 \cup E_2 \cup E_3 \cup E_4$, the assertion follows from Lemma 7.1.10 [d).]. If $P \in (L_{12} \cup L_{13} \cup L_{14} \cup L_{23} \cup L_{24} \cup L_{34}) \backslash (\bigcup_{i\in\{1,2,3,4\}} E_i)$, the assertion follows from Lemma 7.1.9 [a).]. If P is a general point, the assertion follows from Lemma 7.1.1.

V. If $P \in E_1 \cup E_1'$, the assertion follows from Lemma 7.1.15 [a).]. If $P \in \bigcup_{i\in\{1,2,3\}} \left(L_{1,i} \cup L_{1,i}'\right) \backslash (E_1 \cup E_1')$, the assertion follows from Lemma 7.1.4 [a).]. If P is a general point, the assertion follows from Lemma 7.1.1.

VI. If $P \in E_1'$, the assertion follows from Lemma 7.1.15 [a).]. If $P \in E_1$, the assertion follows from Lemma 7.1.15 [b).]. If $P \in \bigcup_{i\in\{1,2,3\}} L_{1,i} \backslash E_1'$, the assertion follows from Lemma 7.1.4 [b).]. If $P \in L_1 \backslash E_1$, the assertion follows from Lemma 7.1.4 [a).]. If $P \in E_2$, the assertion follows from Lemma 7.1.10 [e).]. If $P \in L_{12} \backslash (E_1 \cup E_2)$, the assertion follows from Lemma 7.1.11. If $P \in \left(\bigcup_{i\in\{1,2,3\}} L_{2,i}\right) \backslash \left(E_2 \cup \bigcup_{i\in\{1,2,3\}} L_{1,i}\right)$, the assertion follows from Lemma 7.1.2 [d).]. If P is a general point, the assertion follows from Lemma 7.1.1.

VII. If $P \in E_1 \cup E_1'$, the assertion follows from Lemma 7.1.15 [b).]. If $P \in E_2 \cup E_2'$, the assertion follows from Lemma 7.1.10 [f).]. If $P \in (L_{12} \cup L_{12}') \backslash (E_1 \cup E_1' \cup E_2 \cup E_2')$, the assertion follows from Lemma 7.1.11. If $P \in L_2 \backslash (E_2 \cup E_2')$, the assertion follows from Lemma 7.1.9 [a).]. If $P \in (L_{1,1} \cup L_{1,1}') \backslash (E_1 \cup E_1')$, the assertion follows from Lemma 7.1.4 [b).]. If $P \in (L_{2,1} \cup L_{2,1}') \backslash (E_2 \cup E_2')$, the assertion follows from Lemma 7.1.2 [e).]. If P is a general point, the assertion follows from Lemma 7.1.1.

VIII. If $P \in E_1' \cup E_2'$, the assertion follows from Lemma 7.1.15 [a).]. If $P \in E_1 \cup E_2$, the assertion follows from Lemma 7.1.15 [c).]. If $P \in L_{12} \backslash (E_1 \cup E_2)$, the assertion follows from Lemma 7.1.16 [a).]. If $P \in \left(\bigcup_{i\in\{1,2\},j\in\{1,2,3\}} L_{i,j}\right) \backslash (E_1' \cup E_2')$, the assertion follows from Lemma 7.1.4 [c).]. If P is a general point, the assertion follows from Lemma 7.1.1.

IX. If $P \in E_1' \cup E_2'$, the assertion follows from Lemma 7.1.15 [b).]. If $P \in E_1 \cup E_2$, the assertion follows from Lemma 7.1.15 [c).]. If $P \in L_{12} \backslash (E_1 \cup E_2)$, the assertion follows from Lemma 7.1.16 [a).]. If $P \in E_3$, the assertion follows from Lemma 7.1.10 [g).]. If $P \in (L_{13} \cup L_{23}) \backslash (E_1' \cup E_2' \cup E_3)$, the assertion follows from Lemma 7.1.11. If $P \in (L_{1,1} \cup L_{2,1}) \backslash (E_1' \cup E_2')$, the assertion follows from Lemma 7.1.4 [c).]. If P is a general point, the assertion follows from Lemma 7.1.1.

X. If $P \in E_1 \cup E_1' \cup E_2 \cup E_2' \cup E_3 \cup E_3'$, the assertion follows from Lemma 7.1.15 [c).]. If $P \in (L_{12} \cup L_{13} \cup L_{23}) \backslash (E_1 \cup E_1' \cup E_2 \cup E_2' \cup E_3 \cup E_3')$, the assertion follows from Lemma 7.1.16. If P is a general point, the assertion follows from Lemma 7.1.1.

XI. If $P \in E_2$, the assertion follows from Lemma 7.1.22 [a).]. If $P \in (E_1 \cup E_1') \backslash E_2$, the assertion follows from Lemma 7.1.17 [a).]. If $P \in \bigcup_{i \in \{1,2\}} \left(L_{1,i} \cup L_{1,i}'\right) \backslash (E_1 \cup E_1')$, the assertion follows from Lemma 7.1.5. If $P \in L_{2,i} \backslash E_2$, the assertion follows from Lemma 7.1.9 [b).]. If P is a general point, the assertion follows from Lemma 7.1.1.

XII. If $P \in E_2$, the assertion follows from Lemma 7.1.22 [a).]. If $P \in E_1' \backslash E_2$, the assertion follows from Lemma 7.1.17 [a).]. If $P \in E_1 \backslash E_2$, the assertion follows from Lemma 7.1.17 [b).]. If $P \in L_{13} \backslash E_1$, the assertion follows from Lemma 7.1.13 [a).]. If $P \in E_3 \backslash L_{13}$, the assertion follows from Lemma 7.1.10 [f).]. If $P \in L_{2,1} \backslash E_2$, the assertion follows from Lemma 7.1.9 [b).]. If $P \in (L_{1,1} \cup L_{1,1}') \backslash E_1'$, the assertion follows from Lemma 7.1.5. If $P \in (L_{3,1} \cup L_{3,1}') \backslash (E_3 \cup L_{1,1} \cup L_{1,1}')$, the assertion follows from Lemma 7.1.2 [f).]. If P is a general point, the assertion follows from Lemma 7.1.1.

XIII. If $P \in E_2$, the assertion follows from Lemma 7.1.22 [a).]. If $P \in (E_1 \cup E_1') \backslash E_2$, the assertion follows from Lemma 7.1.17 [b).]. If $P \in (L_{13} \cup L_{13}') \backslash (E_1 \cup E_1')$, the assertion follows from Lemma 7.1.13 [a).]. If $P \in L_{2,1} \backslash E_2$, the assertion follows from Lemma 7.1.9 [b).]. If $P \in L_3 \backslash (E_3 \cup E_3')$, the assertion follows from Lemma 7.1.9 [a).]. If $P \in (E_3 \cup E_3') \backslash (L_{13} \cup L_{13}')$, the assertion follows from Lemma 7.1.10 [f).]. If P is a general point, the assertion follows from Lemma 7.1.1.

XIV. If $P \in E_1$, the assertion follows from Lemma 7.1.26. If $P \in E_2 \backslash E_1$, the assertion follows from Lemma 7.1.23. If $P \in E_3 \backslash E_1$, the assertion follows from Lemma 7.1.18. If $P \in E_4 \backslash E_2$, the assertion follows from Lemma 7.1.20 [a).]. If $P \in L_{1,1} \backslash E_1$, the assertion follows from Lemma 7.1.12. If $P \in L_{3,1} \backslash E_3$, the assertion follows from Lemma 7.1.8 [a).]. If $P \in (L_{4,1} \cup L_{4,2}) \backslash E_4$, the assertion follows from Lemma 7.1.6. If P is a general point, the assertion follows from Lemma 7.1.1.

XV. If $P \in E_1$, the assertion follows from Lemma 7.1.26. If $P \in E_2 \backslash E_1$, the assertion follows from Lemma 7.1.23. If $P \in E_3 \backslash E_1$, the assertion follows from Lemma 7.1.18. If $P \in E_4 \backslash E_2$, the assertion follows from Lemma 7.1.20 [b).]. If $P \in L_{1,1} \backslash E_1$, the assertion follows from Lemma 7.1.12. If $P \in L_{45} \backslash E_4$, the assertion follows from Lemma 7.1.14. If $P \in E_5 \backslash L_{45}$, the assertion follows from Lemma 7.1.10 [j).]. If $P \in L_{3,1} \backslash E_3$, the assertion follows from Lemma 7.1.8 [b).]. If $P \in L_{5,1} \backslash (L_{3,1} \cup E_5)$, the assertion follows from Lemma 7.1.10 [g).]. If P is a general point, the assertion follows from Lemma 7.1.1.

XVI. If $P \in E_2$, the assertion follows from Lemma 7.1.29 [a).]. If $P \in E_3 \backslash E_2$, the assertion follows from Lemma 7.1.27. If $P \in E_4 \backslash E_3$, the assertion follows from Lemma 7.1.25 [a).]. If $P \in E_5 \backslash E_4$, the assertion follows from Lemma 7.1.21 [a).]. If $P \in E_1 \backslash E_2$, the assertion follows from Lemma 7.1.19. If $P \in L_{2,1} \backslash E_2$, the assertion follows from Lemma 7.1.16. If $P \in (L_{5,1} \cup L_{5,2}) \backslash E_5$, the assertion follows from Lemma 7.1.7. If P is a general point, the assertion follows from Lemma 7.1.1.

XVII. If $P \in E_2$, the assertion follows from Lemma 7.1.29 [a).]. If $P \in E_3 \backslash E_2$, the assertion follows from Lemma 7.1.27. If $P \in E_4 \backslash E_3$, the assertion follows from Lemma 7.1.25 [a).]. If $P \in E_5 \backslash E_4$, the assertion follows from Lemma 7.1.21 [b).]. If $P \in E_1 \backslash E_2$, the assertion follows from Lemma 7.1.19. If $P \in L_{2,1} \backslash E_2$, the assertion follows from Lemma 7.1.16 [b).]. If $P \in L_{56} \backslash E_5$, the assertion follows from Lemma 7.1.16 [c).]. If $P \in E_6 \backslash L_{56}$, the assertion follows from Lemma 7.1.10 [k).]. If P is a general point, the assertion follows from Lemma 7.1.1.

XVIII. If $P \in E$, the assertion follows from Lemma 7.1.30 [a).]. If $P \in (E_1 \cup E_2 \cup E_3) \backslash E$, the assertion follows from Lemma 7.1.22 [b).]. If $P \in (L_{1,1} \cup L_{2,1} \cup L_{3,1}) \backslash (E_1 \cup E_2 \cup E_3)$, the assertion follows from Lemma 7.1.9 [c).]. If P is a general point, the assertion follows from Lemma 7.1.1.

XIX. If $P \in E_2$, the assertion follows from Lemma 7.1.31. If $P \in E_3 \backslash E_2$, the assertion follows from Lemma 7.1.30 [b).]. If $P \in E \backslash E_2$, the assertion follows from Lemma 7.1.28. If $P \in E_1 \backslash E_2$, the assertion follows from Lemma 7.1.24. If $P \in E_4 \backslash E_3$, the assertion follows from Lemma 7.1.22 [c).]. If $P \in L \backslash E$, the assertion follows from Lemma 7.1.13 [b).] If $P \in L_{4,1} \backslash E_4$, the assertion follows from Lemma 7.1.9 [d).]. If P is a general point, the assertion follows from Lemma 7.1.1.

XX. If $P \in E_3$, the assertion follows from Lemma 7.1.34. If $P \in E_4 \backslash E_3$, the assertion follows from Lemma 7.1.33. If $P \in E_2 \backslash E_3$, the assertion follows from Lemma 7.1.32. If $P \in E \backslash E_3$, the assertion follows from Lemma 7.1.30 [c).]. If $P \in E_5 \backslash E_4$, the assertion follows from Lemma 7.1.29 [b).]. If $P \in E_1 \backslash E_2$, the assertion follows from Lemma 7.1.25 [b).]. If $P \in L_{5,1} \backslash E_5$, the assertion follows from Lemma 7.1.16 [d).]. If P is a general point, the assertion follows from Lemma 7.1.1.

□

Chapter 8

Du Val del Pezzo Surfaces of Degree 2

In (Araujo et al., 2023, Lemma 2.15) it was proven that $\delta(X) = \frac{9}{5}$ when X is a smooth del Pezzo surface of degree 2 and $|-K_X|$ contains a tacnodal curve, and $\delta(X) = \frac{15}{8}$ when X is a smooth del Pezzo surface of degree 2 and $|-K_X|$ does not contain a tacnodal curve. In this section we compute δ-invariants of Du Val del Pezzo surfaces of degree 2.

MAIN THEOREM Let X be the Du Val del Pezzo surface of degree 2. Then the δ-invariant of X is uniquely determined by the degree of X, the number of lines on X, and the type of singularities on X which is given in the following table:

K_X^2	# lines	Sing(X)	$\delta(X)$
2	44	$\mathbb{A}_1$	$\frac{3}{2}$
2	34	$2\mathbb{A}_1$	$\frac{3}{2}$
2	26	$3\mathbb{A}_1$	$\frac{3}{2}$
2	25	$3\mathbb{A}_1$	$\frac{3}{2}$
2	20	$4\mathbb{A}_1$	$\frac{3}{2}$
2	19	$4\mathbb{A}_1$	$\frac{3}{2}$
2	14	$5\mathbb{A}_1$	$\frac{3}{2}$
2	10	$6\mathbb{A}_1$	$\frac{3}{2}$
2	31	$\mathbb{A}_2$	$\frac{6}{5}$
2	20	$\mathbb{A}_2+\mathbb{A}_1$	$\frac{6}{5}$
2	18	$\mathbb{A}_2+2\mathbb{A}_1$	$\frac{6}{5}$
2	13	$\mathbb{A}_2+3\mathbb{A}_1$	$\frac{6}{5}$
2	16	$2\mathbb{A}_2$	$\frac{6}{5}$
2	12	$2\mathbb{A}_2+\mathbb{A}_1$	$\frac{6}{5}$
2	8	$3\mathbb{A}_2$	$\frac{6}{5}$
2	22	$\mathbb{A}_3$	1
2	16	$\mathbb{A}_3+\mathbb{A}_1$	1
2	15	$\mathbb{A}_3+\mathbb{A}_1$	1
2	12	$\mathbb{A}_3+2\mathbb{A}_1$	1
2	11	$\mathbb{A}_3+2\mathbb{A}_1$	1
2	8	$\mathbb{A}_3+3\mathbb{A}_1$	1
2	10	$\mathbb{A}_3+\mathbb{A}_2$	1
2	7	$\mathbb{A}_3+\mathbb{A}_2+\mathbb{A}_1$	1
2	6	$2\mathbb{A}_3$	1
2	4	$2\mathbb{A}_3+\mathbb{A}_1$	1
2	14	$\mathbb{A}_4$	$\frac{12}{13}$
2	10	$\mathbb{A}_4+\mathbb{A}_1$	$\frac{12}{13}$
2	6	$\mathbb{A}_4+\mathbb{A}_2$	$\frac{12}{13}$
2	8	$\mathbb{A}_5$	$\frac{6}{7}$
2	7	$\mathbb{A}_5$	$\frac{3}{4}$
2	6	$\mathbb{A}_5+\mathbb{A}_1$	$\frac{6}{7}$
2	5	$\mathbb{A}_5+\mathbb{A}_1$	$\frac{3}{4}$
2	3	$\mathbb{A}_5+\mathbb{A}_2$	$\frac{3}{4}$
2	4	$\mathbb{A}_6$	$\frac{4}{5}$
2	2	$\mathbb{A}_7$	$\frac{3}{4}$
2	14	$\mathbb{D}_4$	$\frac{3}{4}$
2	9	$\mathbb{D}_4+\mathbb{A}_1$	$\frac{3}{4}$
2	6	$\mathbb{D}_4+2\mathbb{A}_1$	$\frac{3}{4}$
2	4	$\mathbb{D}_4+3\mathbb{A}_1$	$\frac{3}{4}$
2	8	$\mathbb{D}_5$	$\frac{3}{5}$
2	5	$\mathbb{D}_5+\mathbb{A}_1$	$\frac{3}{5}$
2	3	$\mathbb{D}_6$	$\frac{1}{2}$
2	2	$\mathbb{D}_6+\mathbb{A}_1$	$\frac{1}{2}$
2	4	$\mathbb{E}_6$	$\frac{3}{7}$
2	1	$\mathbb{E}_7$	$\frac{3}{10}$

Table 8.2: δ-invariants of Du Val del Pezzo surfaces of degree 2

8.1 General results for degree 2

Let X be a del Pezzo surface of degree 2 with at most Du Val singularities. Let S be a weak resolution of X. We will call an image of a (-1)-curve in S on X **a line** as was done in Cheltsov and Prokhorov (2021).

Lemma 8.1.1. *Assume that the point Q is not contained in any line that passes through a singular point of X. Then $\delta_Q(X) \geq \frac{9}{5}$.*

Proof. Follows from Remark (Araujo et al., 2023, 2.14) and proof of Lemma (Araujo et al., 2023, 2.15). □

Now we consider a curve A on S. Small circles correspond to a (-1)-curves and large circles correspond to a (-2)-curves on dual graphs.

Lemma 8.1.2. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

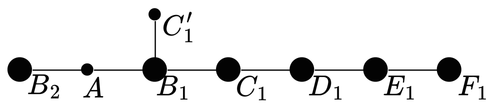


Figure 8.1: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{45}{16}$

Then $\tau(A) = \frac{4}{3}$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{6}(5B_1 + 4C_1 + 3D_1 + 2E_1 + F_1 + 3B_2) \text{ if } v \in \left[0, \frac{6}{5}\right], \\ -K_S - vA - \frac{v}{2}B_2 - (v-1)(5B_1 + 4C_1 + 3D_1 + 2E_1 + F_1) - (3v-4)C_1' \text{ if } v \in \left[\frac{6}{5}, \frac{4}{3}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(5B_1 + 4C_1 + 3D_1 + 2E_1 + F_1 + 3B_2) \text{ if } v \in \left[0, \frac{6}{5}\right], \\ \frac{v}{2}B_2 + (v-1)(5B_1 + 4C_1 + 3D_1 + 2E_1 + F_1) + (3v-4)C_1' \text{ if } v \in \left[\frac{6}{5}, \frac{4}{3}\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2 - 2v + \frac{v^2}{3} \text{ if } v \in \left[0, \frac{6}{5}\right], \\ \frac{(4-3v)^2}{2} \text{ if } v \in \left[\frac{6}{5}, \frac{4}{3}\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} 1 - \frac{v}{3} \text{ if } v \in \left[0, \frac{6}{5}\right], \\ 3(2 - \frac{3v}{2}) \text{ if } v \in \left[\frac{6}{5}, \frac{4}{3}\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{45}{26}$ if $P \in A \backslash (B_1 \cup B_2)$.

Proof. The Zariski Decomposition follows from

$$-K_S - vA \sim_{\mathbb{R}} \left(\frac{4}{3} - v\right)A + \frac{1}{3}\left(2B_2 + 5B_1 + 4C_1 + 3D_1 + 2E_1 + F_1 + 2C_1'\right).$$

We have $S_S(A) = \frac{26}{45}$. Thus, $\delta_P(S) \leq \frac{45}{26}$ for $P \in E_2$. Note that for $P \in A \backslash (B_1 \cup B_2)$ we have:

$$h(v) \leq \begin{cases} \frac{(3-v)^2}{18} \text{ if } v \in \left[0, \frac{6}{5}\right], \\ \frac{9(4-3v)^2}{8} \text{ if } v \in \left[\frac{6}{5}, \frac{4}{3}\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{2}{5} < \frac{26}{45}$. Thus, $\delta_P(S) = \frac{45}{26}$ if $P \in A \backslash (B_1 \cup B_2)$. □

Lemma 8.1.3. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph and no other (-2)-curves intersect A:*

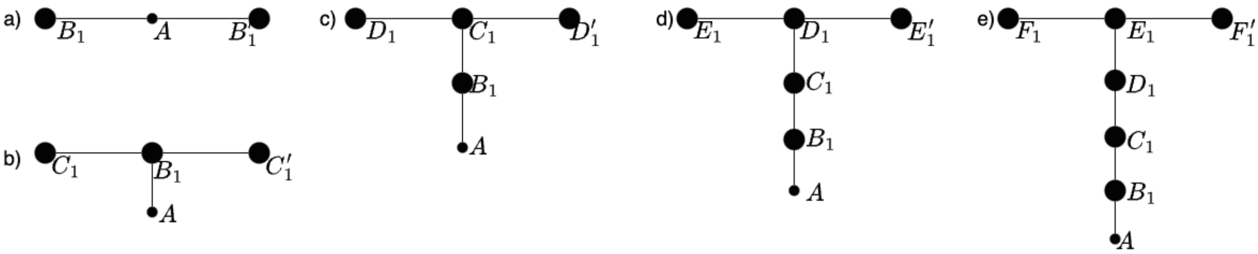


Figure 8.2: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = 2$

Then $\tau(A) = 1$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).
$$P(v) = -K_S - vA - \frac{v}{2}(B_1 + B_1') \text{ if } v \in [0,1].$$
$$N(v) = \frac{v}{2}(B_1 + B_1') \text{ if } v \in [0,1].$$

b).
$$P(v) = -K_S - vA - \frac{v}{2}(2B_1 + C_1 + C_1') \text{ if } v \in [0,1].$$
$$N(v) = \frac{v}{2}(2B_1 + C_1 + C_1') \text{ if } v \in [0,1].$$

c).
$$P(v) = -K_S - vA - \frac{v}{2}(2B_1 + 2C_1 + D_1 + D_1') \text{ if } v \in [0,1].$$
$$N(v) = \frac{v}{2}(2B_1 + 2C_1 + D_1 + D_1') \text{ if } v \in [0,1].$$

d).
$$P(v) = -K_S - vA - \frac{v}{2}(2B_1 + 2C_1 + 2D_1 + E_1 + E_1') \text{ if } v \in [0,1].$$
$$N(v) = \frac{v}{2}(2B_1 + 2C_1 + 2D_1 + E_1 + E_1') \text{ if } v \in [0,1].$$

e).
$$P(v) = -K_S - vA - \frac{v}{2}(2B_1 + 2C_1 + 2D_1 + 2E_1 + F_1 + F_1') \text{ if } v \in [0,1].$$
$$N(v) = \frac{v}{2}(2B_1 + 2C_1 + 2D_1 + 2E_1 + F_1 + F_1') \text{ if } v \in [0,1].$$

Moreover,

$$(P(v))^2 = 2(1-v) \text{ and } P(v) \cdot A = 1 \text{ if } v \in [0,1].$$

In this case: $\delta_P(S) = 2$ if $P \in A \backslash (B_1 \cup B_1')$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim (1-v)A + A' + B_1 + B_2$ where A' is the image of A under Geiser involution. Similar statement holds in other parts. We have $S_S(A) = \frac{1}{2}$. Thus, $\delta_P(S) \leq 2$ for $P \in A$. Note that for $P \in A \backslash (B_1 \cup B_1')$ we have $h(v) = 1/2$ if $v \in [0,1]$. So for these points $S(W^A_{\bullet,\bullet}; P) = \frac{1}{2}$. Thus, $\delta_P(S) = 2$ if $P \in A \backslash (B_1 \cup B_1')$. □

Lemma 8.1.4. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

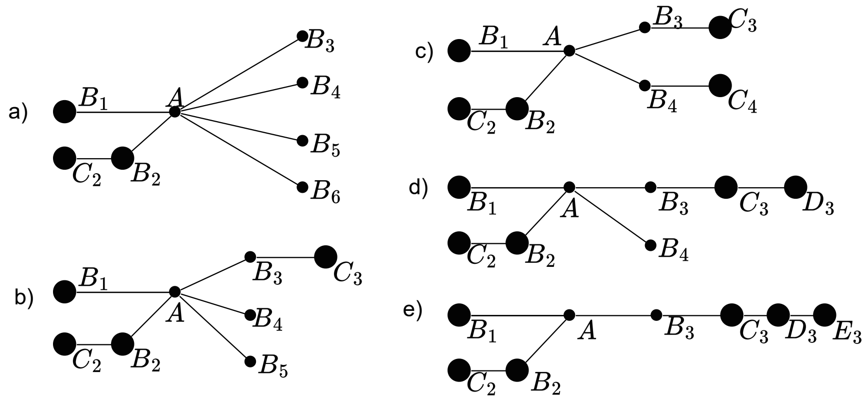


Figure 8.3: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{15}{8}$

Then $\tau(A) = \frac{6}{5}$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{6}(3B_1 + 4B_2 + 2C_2) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{6}(3B_1 + 4B_2 + 2C_2) - (v-1)(B_3 + B_4 + B_5 + B_6) \text{ if } v \in \left[1, \frac{6}{5}\right]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{6}(3B_1 + 4B_2 + 2C_2) \text{ if } v \in [0,1], \\ \frac{v}{6}(3B_1 + 4B_2 + 2C_2) + (v-1)(B_3 + B_4 + B_5 + B_6) \text{ if } v \in \left[1, \frac{6}{5}\right]. \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{6}(3B_1 + 4B_2 + 2C_2) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{6}(3B_1 + 4B_2 + 2C_2) - (v-1)(2B_3 + C_3 + B_4 + B_5) \text{ if } v \in \left[1, \frac{6}{5}\right]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{6}(3B_1 + 4B_2 + 2C_2) \text{ if } v \in [0,1], \\ \frac{v}{6}(3B_1 + 4B_2 + 2C_2) + (v-1)(2B_3 + C_3 + B_4 + B_5) \text{ if } v \in \left[1, \frac{6}{5}\right]. \end{cases}$$

c).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{6}(3B_1 + 4B_2 + 2C_2) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{6}(3B_1 + 4B_2 + 2C_2) - (v-1)(2B_3 + C_3 + 2B_4 + C_4) \text{ if } v \in \left[1, \frac{6}{5}\right]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{6}(3B_1 + 4B_2 + 2C_2) \text{ if } v \in [0,1], \\ \frac{v}{6}(3B_1 + 4B_2 + 2C_2) + (v-1)(2B_3 + C_3 + 2B_4 + C_4) \text{ if } v \in \left[1, \frac{6}{5}\right]. \end{cases}$$

d).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{6}(3B_1 + 4B_2 + 2C_2) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{6}(3B_1 + 4B_2 + 2C_2) - (v-1)(3B_3 + 2C_3 + D_3 + B_4) \text{ if } v \in \left[1, \frac{6}{5}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(3B_1+4B_2+2C_2) \text{ if } v\in[0,1], \\ \frac{v}{6}(3B_1+4B_2+2C_2)+(v-1)(3B_3+2C_3+D_3+B_4) \text{ if } v\in\left[1,\frac{6}{5}\right]. \end{cases}$$

e). $$P(v) = \begin{cases} -K_S-vA-\frac{v}{6}(3B_1+4B_2+2C_2) \text{ if } v\in[0,1], \\ -K_S-vA-\frac{v}{6}(3B_1+4B_2+2C_2)-(v-1)(4B_3+3C_3+2D_3+E_3) \text{ if } v\in\left[1,\frac{6}{5}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(3B_1+4B_2+2C_2) \text{ if } v\in[0,1], \\ \frac{v}{6}(3B_1+4B_2+2C_2)+(v-1)(4B_3+3C_3+2D_3+E_3) \text{ if } v\in\left[1,\frac{6}{5}\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2-2v+\frac{v^2}{6} \text{ if } v\in[0,1], \\ \frac{(5v-6)^2}{6} \text{ if } v\in\left[1,\frac{6}{5}\right]. \end{cases} \quad P(v)\cdot A = \begin{cases} 1-\frac{v}{6} \text{ if } v\in[0,1], \\ 5(1-\frac{5v}{6}) \text{ if } v\in\left[1,\frac{6}{5}\right]. \end{cases}$$

In this case for $P\in A\backslash(B_1\cup B_2)$ we have: $\delta_P(S)=\frac{15}{8}$.

Proof. The Zariski Decomposition in part a). follows from

$$-K_S-vA \sim_{\mathbb{R}} \Big(\frac{6}{5}-v\Big)A+\frac{3}{5}B_1+\frac{2}{5}\Big(2B_2+C_2\Big)+\frac{1}{5}\Big(B_3+B_4+B_5+B_6\Big).$$

A similar statement holds in other parts. We have $S_S(A)=\frac{8}{15}$. Thus, $\delta_P(S)\le\frac{15}{8}$ for $P\in A$. Note that for $P\in A\backslash(B_1\cup B_2)$ we have:

$$h(v) \le \begin{cases} \frac{(6-v)^2}{72} \text{ if } v\in[0,1], \\ \frac{5(6-5v)(23v-18)}{72} \text{ if } v\in\left[1,\frac{6}{5}\right]. \end{cases}$$

So we have $S(W^A_{\bullet,\bullet};P)\le\frac{7}{15}<\frac{8}{15}$. Thus, $\delta_P(S)=\frac{15}{8}$ if $P\in A\backslash(B_1\cup B_2)$. □

Lemma 8.1.5. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

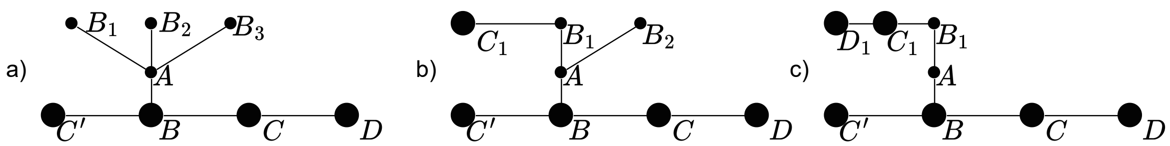


Figure 8.4: Dual graph: $(-K_S)^2=2$ and $\delta_P(S)=\frac{24}{13}$

Then $\tau(A)=\frac{5}{4}$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

a). $$P(v) = \begin{cases} -K_S-vA-\frac{v}{5}(3C'+6B+4C+2D) \text{ if } v\in[0,1], \\ -K_S-vA-\frac{v}{5}(3C'+6B+4C+2D)-(v-1)(B_1+B_2+B_3) \text{ if } v\in\left[1,\frac{5}{4}\right]. \end{cases}$$

$$N(v)=\begin{cases}\frac{v}{5}(3C'+6B+4C+2D) \text{ if } v\in[0,1],\\ \frac{v}{5}(3C'+6B+4C+2D)+(v-1)(B_1+B_2+B_3) \text{ if } v\in\left[1,\frac{5}{4}\right].\end{cases}$$

b). $$P(v)=\begin{cases}-K_S-vA-\frac{v}{5}(3C'+6B+4C+2D) \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{5}(3C'+6B+4C+2D)-(v-1)(2B_1+C_1+B_2) \text{ if } v\in\left[1,\frac{5}{4}\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{5}(3C'+6B+4C+2D) \text{ if } v\in[0,1],\\ \frac{v}{5}(3C'+6B+4C+2D)+(v-1)(2B_1+C_1+B_2) \text{ if } v\in\left[1,\frac{5}{4}\right].\end{cases}$$

c). $$P(v)=\begin{cases}-K_S-vA-\frac{v}{5}(3C'+6B+4C+2D) \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{5}(3C'+6B+4C+2D)-(v-1)(3B_1+2C_1+D_1) \text{ if } v\in\left[1,\frac{5}{4}\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{5}(3C'+6B+4C+2D) \text{ if } v\in[0,1],\\ \frac{v}{5}(3C'+6B+4C+2D)+(v-1)(3B_1+2C_1+D_1) \text{ if } v\in\left[1,\frac{5}{4}\right].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}2-2v+\frac{v^2}{5} \text{ if } v\in[0,1],\\ \frac{(5-4v)^2}{5} \text{ if } v\in\left[1,\frac{5}{4}\right].\end{cases}\qquad P(v)\cdot A=\begin{cases}1-\frac{v}{5} \text{ if } v\in[0,1],\\ 4(1-\frac{4v}{5}) \text{ if } v\in\left[1,\frac{5}{4}\right].\end{cases}$$

In this case $\delta_P(S)=\frac{24}{13}$ if $P\in A\backslash B$.

Proof. In part a). the Zariski Decomposition follows from

$$-K_S-vA\sim_{\mathbb{R}}\Big(\frac{5}{4}-v\Big)A+\frac{1}{4}\Big(3C'+6B+4C+2D+B_1+B_2+B_3\Big).$$

A similar statement holds in other parts. We have $S_S(A)=\frac{13}{24}$. Thus, $\delta_P(S)\leq\frac{24}{13}$ for $P\in A$. Note that for $P\in A\backslash B$ we have:

$$h(v)\leq\begin{cases}\frac{(5-v)^2}{50} \text{ if } v\in[0,1],\\ \frac{4(5-4v)(7v-5)}{25} \text{ if } v\in\left[1,\frac{5}{4}\right].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\leq\frac{11}{24}<\frac{13}{24}$. Thus, $\delta_P(S)=\frac{24}{13}$ if $P\in A\backslash B$. □

Lemma 8.1.6. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

Then $\tau(A)=\frac{4}{3}$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

a). $$P(v)=\begin{cases}-K_S-vA-\frac{v}{4}(3B_2+2C_2+D_2+2B_1) \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{4}(3B_2+2C_2+D_2+2B_1)-(v-1)(B_1+B_2) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$

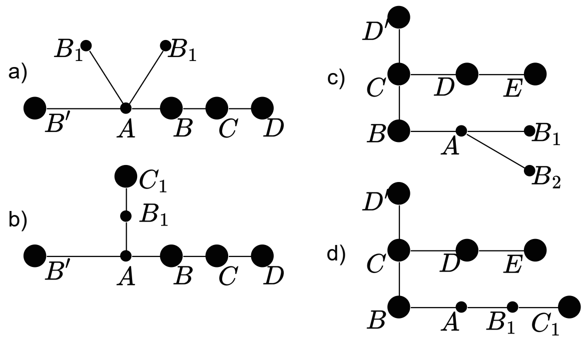


Figure 8.5: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{9}{5}$

$$N(v) = \begin{cases} \frac{v}{4}(3B_2+2C_2+D_2+2B_1) \textit{ if } v \in [0,1], \\ \frac{v}{4}(3B_2+2C_2+D_2+2B_1)+(v-1)(B_1+B_2) \textit{ if } v \in \left[1,\frac{4}{3}\right]. \end{cases}$$

b). $$P(v) = \begin{cases} -K_S - vA - \frac{v}{4}(3B_2+2C_2+D_2+2B_1) \textit{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{4}(3B_2+2C_2+D_2+2B_1)-(v-1)(2B_1+C_1) \textit{ if } v \in \left[1,\frac{4}{3}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(3B_2+2C_2+D_2+2B_1) \textit{ if } v \in [0,1], \\ \frac{v}{4}(3B_2+2C_2+D_2+2B_1)+(v-1)(2B_1+C_1) \textit{ if } v \in \left[1,\frac{4}{3}\right]. \end{cases}$$

c). $$P(v) = \begin{cases} -K_S - vA - \frac{v}{4}(5B+6C+4D+2E+3D') \textit{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{4}(5B+6C+4D+2E+3D')-(v-1)(B_1+B_2) \textit{ if } v \in \left[1,\frac{4}{3}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(5B+6C+4D+2E+3D') \textit{ if } v \in [0,1], \\ \frac{v}{4}(5B+6C+4D+2E+3D')+(v-1)(B_1+B_2) \textit{ if } v \in \left[1,\frac{4}{3}\right]. \end{cases}$$

d). $$P(v) = \begin{cases} -K_S - vA - \frac{v}{4}(5B+6C+4D+2E+3D') \textit{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{4}(5B+6C+4D+2E+3D')-(v-1)(B_1+C_1) \textit{ if } v \in \left[1,\frac{4}{3}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(5B+6C+4D+2E+3D') \textit{ if } v \in [0,1], \\ \frac{v}{4}(5B+6C+4D+2E+3D')+(v-1)(2B_1+C_1) \textit{ if } v \in \left[1,\frac{4}{3}\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2-2v+\frac{v^2}{4} \textit{ if } v \in [0,1], \\ \frac{(4-3v)^2}{4} \textit{ if } v \in \left[1,\frac{4}{3}\right]. \end{cases} \quad P(v)\cdot A = \begin{cases} 1+\frac{v}{4} \textit{ if } v \in [0,1], \\ 3(1-\frac{3v}{4}) \textit{ if } v \in \left[1,\frac{4}{3}\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{9}{5}$ *if* $P \in A \backslash (B \cup B')$.

Proof. The Zariski Decomposition in part a). follows from

$$-K_S - vA \sim_{\mathbb{R}} \Big(\frac{4}{3}-v\Big)A + \frac{1}{3}\Big(2B'+3B+2C+D+B_1+B_2\Big).$$

A similar statement holds in other parts. We have $S_S(A) = \frac{5}{9}$. Thus, $\delta_P(S) \leq \frac{9}{5}$ for $P \in A$. Note that for $P \in A \backslash (B \cup B')$ we have:

$$h(v) \leq \begin{cases} \frac{(4+v)^2}{32} \text{ if } v \in [0,1], \\ \frac{3(4-3v)(7v-4)}{32} \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{4}{9} < \frac{5}{9}$. Thus, $\delta_P(S) = \frac{9}{5}$ if $P \in A \backslash (B_1 \cup B_2)$. □

Lemma 8.1.7. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

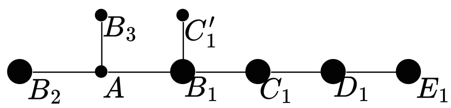


Figure 8.6: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{72}{41}$

Then $\tau(A) = \frac{4}{3}$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) - \frac{v}{2}B_2 \textit{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) - \frac{v}{2}B_2 - (v-1)B_3 \textit{ if } v \in \left[1, \frac{5}{4}\right], \\ -K_S - vA - (v-1)(4B_1 + 3C_1 + 2D_1 + E_1 + B_3) - \frac{v}{2}B_2 - (4v-5)C_1' \textit{ if } v \in \left[\frac{5}{4}, \frac{4}{3}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) + \frac{v}{2}B_2 \textit{ if } v \in [0,1], \\ \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) + \frac{v}{2}B_2 + (v-1)B_3 \textit{ if } v \in \left[1, \frac{5}{4}\right], \\ (v-1)(4B_1 + 3C_1 + 2D_1 + E_1 + B_3) + \frac{v}{2}B_2 + (4v-5)C_1' \textit{ if } v \in \left[\frac{5}{4}, \frac{4}{3}\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} \frac{3v^2}{10} - 2v + 2 \textit{ if } v \in [0,1], \\ \frac{13v^2}{10} - 4v + 3 \textit{ if } v \in \left[1, \frac{5}{4}\right], \\ \frac{(4-3v)^2}{2} \textit{ if } v \in \left[\frac{5}{4}, \frac{4}{3}\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} 1 - \frac{3v}{10} \textit{ if } v \in [0,1], \\ 2 - \frac{13v}{10} \textit{ if } v \in \left[1, \frac{5}{4}\right], \\ 3(2 - \frac{3v}{2}) \textit{ if } v \in \left[\frac{5}{4}, \frac{4}{3}\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{72}{41}$ if $P \in A \backslash (B_1 \cup B_2)$.

Proof. In part a). the Zariski Decomposition follows from

$$-K_S - vA \sim_{\mathbb{R}} \left(\frac{4}{3} - v\right)A + \frac{1}{3}\left(4B_1 + 3C_1 + 2D_1 + E_1 + B_3 + 2B_2 + C_1'\right).$$

A similar statement holds in other parts. We have $S_S(A) = \frac{41}{72}$. Thus, $\delta_P(S) \le \frac{72}{11}$ for $P \in A$. Note that for $P \in A \backslash (B_1 \cup B_2)$ we have:

$$h(v) \le \begin{cases} \frac{(10-3v)^2}{200} \text{ if } v \in [0,1], \\ \frac{7(20-13v)v}{200} \text{ if } v \in \left[1, \frac{5}{4}\right], \\ \frac{3(4-3v)(8-5v)}{8} \text{ if } v \in \left[\frac{5}{4}, \frac{4}{3}\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \le \frac{61}{144} < \frac{41}{72}$. Thus, $\delta_P(S) = \frac{72}{41}$ if $P \in A \backslash (B_1 \cup B_2)$. □

Lemma 8.1.8. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

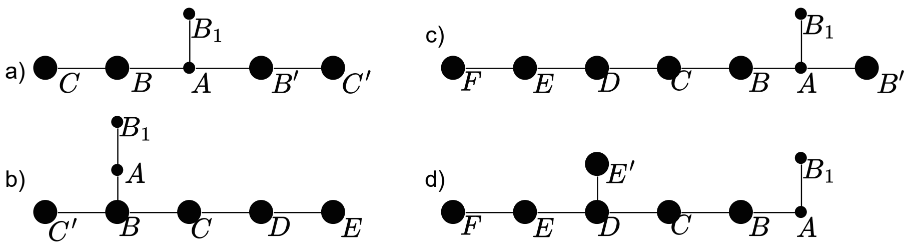


Figure 8.7: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{12}{7}$

Then $\tau(A) = \frac{3}{2}$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(C + 2B + 2B' + C') \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{3}(C + 2B + 2B' + C') - (v-1)B_1 \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(C + 2B + 2B' + C') \text{ if } v \in [0,1], \\ \frac{v}{3}(C + 2B + 2B' + C') + (v-1)B_1 \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

b).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2C' + 4B + 3C + 2D + E) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{3}(2C' + 4B + 3C + 2D + E) - (v-1)B_1 \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2C' + 4B + 3C + 2D + E) \text{ if } v \in [0,1], \\ \frac{v}{3}(2C' + 4B + 3C + 2D + E) + (v-1)B_1 \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

c).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{6}(5B + 4C + 3D + 2E + F + 3B') \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{6}(5B + 4C + 3D + 2E + F + 3B') - (v-1)B_1 \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(5B + 4C + 3D + 2E + F + 3B') \text{ if } v \in [0,1], \\ \frac{v}{6}(5B + 4C + 3D + 2E + F + 3B') + (v-1)B_1 \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

d).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2F + 4E + 6D + 5C + 4B + 3E') \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{3}(2F + 4E + 6D + 5C + 4B + 3E') - (v-1)B_1 \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2F+4E+6D+5C+4B+3E') \text{ if } v \in [0,1], \\ \frac{v}{6}(2F+4E+6D+5C+4B+3E') + (v-1)B_1 \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2-2v+\frac{v^2}{3} \text{ if } v \in [0,1], \\ \frac{(3-2v)^2}{3} \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases} \quad P(v)\cdot A = \begin{cases} 1-\frac{v}{3} \text{ if } v \in [0,1], \\ 2(1-\frac{2v}{3}) \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{12}{7}$ if $P \in A\backslash(B \cup B')$.

Proof. In part a). the Zariski Decomposition follows from

$$-K_S - vA \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)A + \frac{1}{2}\Big(C + 2B + 2B' + C' + B_1\Big).$$

A similar statement holds in other parts. We have $S_S(A) = \frac{7}{12}$. Thus, $\delta_P(S) \le \frac{12}{7}$ for $P \in A$. Note that for $P \in A\backslash(B \cup B')$ we have:

$$h(v) \le \begin{cases} \frac{(3-v)^2}{18} \text{ if } v \in [0,1], \\ \frac{2v(3-2v)}{9} \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \le \frac{5}{12} < \frac{7}{12}$. Thus, $\delta_P(S) = \frac{12}{7}$ if $P \in A\backslash(B \cup B')$. □

Lemma 8.1.9. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

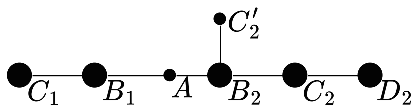


Figure 8.8: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{18}{11}$

Then $\tau(A) = \frac{3}{2}$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2B_1+C_1) - \frac{v}{4}(3B_2+2C_2+D_2) \text{ if } v \in \left[0, \frac{4}{3}\right], \\ -K_S - vA - \frac{v}{3}(2B_1+C_1) - (v-1)(3B_2+2C_2+D_2) - (3v-4)C_2' \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2B_1+C_1) + \frac{v}{4}(3B_2+2C_2+D_2) \text{ if } v \in \left[0, \frac{4}{3}\right], \\ \frac{v}{3}(2B_1+C_1) + (v-1)(3B_2+2C_2+D_2) + (3v-4)C_2' \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2-2v+\frac{5v^2}{12}, \text{ if } v \in \left[0, \frac{4}{3}\right], \\ \frac{2(3-2v)^2}{3} \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right]. \end{cases} \quad P(v)\cdot A = \begin{cases} 1-\frac{5v}{12} \text{ if } v \in \left[0, \frac{4}{3}\right], \\ 4(1-\frac{2v}{3}) \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{18}{11}$ *if* $P \in A \backslash (B_1 \cup B_2)$.

Proof. The Zariski Decomposition follows from

$$-K_S - vA \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)A + \frac{1}{2}\Big(2B_1 + C_1 + 3B_2 + 2C_2 + D_2 + C_2'\Big).$$

We have $S_S(A) = \frac{11}{18}$. Thus, $\delta_P(S) \le \frac{18}{11}$ for $P \in A$. Note that for $P \in A \backslash (B_1 \cup B_2)$ we have:

$$h(v) \le \begin{cases} \frac{(12-5v)^2}{288} \text{ if } v \in \big[0, \frac{4}{3}\big], \\ \frac{8(3-2v)^2}{9} \text{ if } v \in \big[\frac{4}{3}, \frac{3}{2}\big]. \end{cases}$$

So we have $S(W^A_{\bullet,\bullet}; P) = \frac{10}{27} < \frac{11}{18}$. Thus, $\delta_P(S) = \frac{18}{11}$ if $P \in A \backslash (B_1 \cup B_2)$. □

Lemma 8.1.10. *Suppose* P *belongs to a* (-1)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

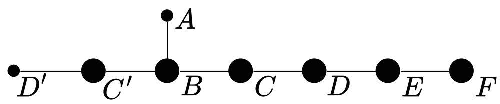


Figure 8.9: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{60}{37}$

Then $\tau(A) = \frac{3}{2}$ *and the Zariski Decomposition of the divisor* $-K_S - vA$ *is given by:*

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{7}(2F + 4E + 6D + 8C + 10B + 5C') \text{ if } v \in \big[0, \frac{7}{5}\big], \\ -K_S - vA - (v-1)(2F + 4E + 6D + 8C + 10B) - (5v-6)C' - (5v-7)D' \text{ if } v \in \big[\frac{7}{5}, \frac{3}{2}\big]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{7}(2F + 4E + 6D + 8C + 10B + 5C') \text{ if } v \in \big[0, \frac{7}{5}\big], \\ (v-1)(2F + 4E + 6D + 8C + 10B) + (5v-6)C' + (5v-7)D' \text{ if } v \in \big[\frac{7}{5}, \frac{3}{2}\big]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2 - 2v + \frac{3v^2}{7} \text{ if } v \in \big[0, \frac{7}{5}\big], \\ (3-2v)^2 \text{ if } v \in \big[\frac{7}{5}, \frac{3}{2}\big]. \end{cases} \quad P(v) \cdot A = \begin{cases} 1 - \frac{3v}{7} \text{ if } v \in \big[0, \frac{7}{5}\big], \\ 2(3-2v) \text{ if } v \in \big[\frac{7}{5}, \frac{3}{2}\big]. \end{cases}$$

In this case $\delta_P(S) = \frac{60}{37}$ *if* $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from

$$-K_S - vA \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)A + \frac{1}{2}\Big(F + 2E + 3D + 4C + 5B + 3C' + D'\Big).$$

We have $S_S(A) = \frac{37}{60}$. Thus, $\delta_P(S) \le \frac{60}{37}$ for $P \in A$. Note that for $P \in A \backslash B$ we have:

$$h(v) \le \begin{cases} \frac{(7-3v)^2}{98} \text{ if } v \in \left[0, \frac{7}{5}\right], \\ 2(3-2v)^2 \text{ if } v \in \left[\frac{7}{5}, \frac{3}{2}\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \le \frac{11}{30} < \frac{37}{60}$. Thus, $\delta_P(S) = \frac{60}{37}$ if $P \in A \backslash B$. □

Lemma 8.1.11. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

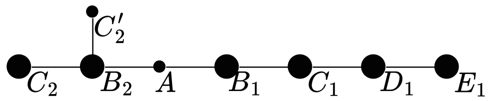


Figure 8.10: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{36}{23}$

Then the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) - \frac{v}{2}(2B_2 + C_2) \textit{ if } v \in \left[0, \frac{3}{2}\right], \\ -K_S - vA - \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) - (v-1)(2B_2 + C_2) - (2v-3)C_2' \textit{ if } v \in \left[\frac{3}{2}, \frac{5}{3}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) + \frac{v}{2}(2B_2 + C_2) \textit{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{v}{5}(4B_1 + 3C_1 + 2D_1 + E_1) + (v-1)(2B_2 + C_2) + (2v-3)C_2' \textit{ if } v \in \left[\frac{3}{2}, \frac{5}{3}\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2 - 2v + \frac{7v^2}{15} \textit{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{(5-3v)^2}{5} \textit{ if } v \in \left[\frac{3}{2}, \frac{5}{3}\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} 1 - \frac{7v}{15} \textit{ if } v \in \left[0, \frac{3}{2}\right], \\ 3(1 - \frac{3v}{5}) \textit{ if } v \in \left[\frac{3}{2}, \frac{5}{3}\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{36}{23}$ if $P \in A \backslash (B_1 \cup B_2)$.

Proof. The Zariski Decomposition follows from

$$-K_S - vA \sim_{\mathbb{R}} \left(\frac{5}{3} - v\right)A + \frac{1}{3}\left(4B_1 + 3C_1 + 2D_1 + E_1 + 4B_2 + 2C_2 + C_2'\right).$$

We have $S_S(A) = \frac{23}{36}$. Thus, $\delta_P(S) \le \frac{36}{23}$ for $P \in A$. Note that for $P \in A \backslash (B_1 \cup B_2)$ we have:

$$h(v) \le \begin{cases} \frac{(15-7v)^2}{450} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{9(5-3v)^2}{50} \text{ if } v \in \left[\frac{3}{2}, \frac{5}{3}\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \le \frac{7}{20} < \frac{23}{36}$. Thus, $\delta_P(S) = \frac{36}{23}$ if $P \in A \backslash (B_1 \cup B_2)$. □

Lemma 8.1.12. *Suppose $P \in A$ where is a (-2)-curve disjoint from other (-2)-curves then $\tau(A) = 1$ and the Zariski decomposition of the divisor $-K_S - vA$ given by:*

$$P(v) = -K_S - vA \text{ and } N(v) = 0 \text{ if } v \in [0,1].$$

Moreover,

$$(P(v))^2 = 2(1-v)(v+1) \text{ and } P(v)\cdot A = 2v \text{ if } v \in [0,1].$$

In this case: $\delta_P(S) = \frac{3}{2}$ if $P \in A$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} L + (1-v)A$ where L is a strict transform of an element $|-K_X|$ passing through a singular point which is the image of A on X. We have $S_S(A) = \frac{2}{3}$. Thus, $\delta_P(S) \le \frac{3}{2}$ for $P \in A$. Note that for $P \in A$ we have $h(v) = 2v^2$ if $v \in [0,1]$. So $S(W^A_{\bullet,\bullet};P) = \frac{2}{3}$. Thus, $\delta_P(S) = \frac{3}{2}$ if $P \in A$. □

Lemma 8.1.13. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

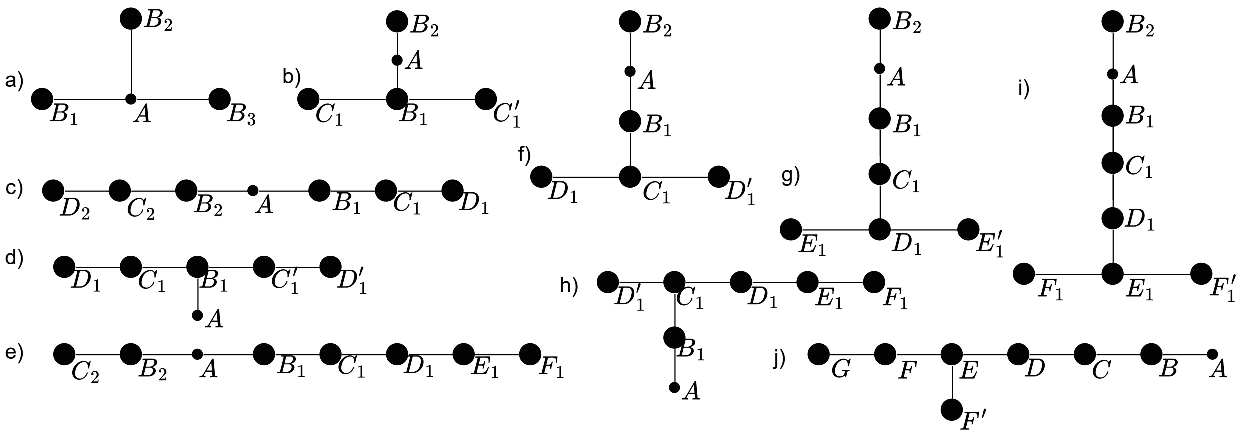


Figure 8.11: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{3}{2}$

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).
$$P(v) = -K_S - vA - \frac{v}{2}(B_1 + B_2 + B_3) \text{ if } v \in [0,2].$$
$$N(v) = \frac{v}{2}(B_1 + B_2 + B_3) \text{ if } v \in [0,2].$$

b).
$$P(v) = -K_S - vA - \frac{v}{2}(2B_1 + C_1 + C_1' + B_2) \text{ if } v \in [0,2].$$
$$N(v) = \frac{v}{2}(2B_1 + C_1 + C_1' + B_2) \text{ if } v \in [0,2].$$

c).
$$P(v) = -K_S - vA - \frac{v}{4}(D_1 + 2C_2 + 3B_1 + 3B_2 + 2C_2 + D_2) \text{ if } v \in [0,2].$$
$$N(v) = \frac{v}{4}(D_1 + 2C_2 + 3B_1 + 3B_2 + 2C_2 + D_2) \text{ if } v \in [0,2].$$

d).
$$P(v) = -K_S - vA - \frac{v}{2}(D_1 + 2C_2 + 3B_1 + 2C_2 + D_2) \text{ if } v \in [0,2].$$

$$N(v)=\frac{v}{2}(D_1+2C_2+3B_1+2C_2+D_2) \text{ if } v\in[0,2].$$

e). $$P(v)=-K_S-vA-\frac{v}{6}(2C_2+4B_2+5B_1+4C_1+3D_1+2E_1+F_1) \text{ if } v\in[0,2].$$
$$N(v)=\frac{v}{6}(2C_2+4B_2+5B_1+4C_1+3D_1+2E_1+F_1) \text{ if } v\in[0,2].$$

f). $$P(v)=-K_S-vA-\frac{v}{2}(2B_1+2C_1+D_1+D_1'+B_2) \text{ if } v\in[0,2].$$
$$N(v)=\frac{v}{2}(2B_1+2C_1+D_1+D_1'+B_2) \text{ if } v\in[0,2].$$

g). $$P(v)=-K_S-vA-\frac{v}{2}(2B_1+2C_1+2D_1+E_1+E_1'+B_2) \text{ if } v\in[0,2].$$
$$N(v)=\frac{v}{2}(2B_1+2C_1+2D_1+E_1+E_1'+B_2) \text{ if } v\in[0,2].$$

h). $$P(v)=-K_S-vA-\frac{v}{2}(3B+2D'+4C+3D+2E+F) \text{ if } v\in[0,2].$$
$$N(v)=\frac{v}{2}(3B+2D'+4C+3D+2E+F) \text{ if } v\in[0,2].$$

i). $$P(v)=-K_S-vA-\frac{v}{2}(2B_1+2C_1+2D_1+2E_1+F_1+F_1'+B_2) \text{ if } v\in[0,2].$$
$$N(v)=\frac{v}{2}(2B_1+2C_1+2D_1+2E_1+F_1+F_1'+B_2) \text{ if } v\in[0,2],$$

j). $$P(v)=-K_S-vA-\frac{v}{2}(2G+4F+6E+5D+4C+3B+3F') \text{ if } v\in[0,2].$$
$$N(v)=\frac{v}{2}(2G+4F+6E+5D+4C+3B+3F') \text{ if } v\in[0,2].$$

Moreover,

$$(P(v))^2=\frac{(2-v)^2}{2}P(v)\cdot A=1-\frac{v}{2} \text{ if } v\in[0,2].$$

In this case $\delta_P(S)=\frac{3}{2}$ *if* $P\in A\backslash(B_1\cup B_2\cup B_3)$.

Proof. The Zariski Decomposition in part a). follows from $-K_S-vA\sim_{\mathbb{R}}(2-v)A+B_1+B_2+B_3$. A similar statement holds in other parts. We have $S_S(A)=\frac{2}{3}$. Thus, $\delta_P(S)\le\frac{3}{2}$ for $P\in A$. Note that for $P\in A\backslash(B_1\cup B_2\cup B_3)$ $h(v)=\frac{(2-v)^2}{8}$ if $v\in[0,2]$. So we have $S(W^A_{\bullet,\bullet};P)=\frac{1}{3}<\frac{2}{3}$. Thus, $\delta_P(S)=\frac{3}{2}$ if $P\in A\backslash(B_1\cup B_2\cup B_3)$. □

Lemma 8.1.14. *Suppose* P *belongs to a* (-1)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

Then $\tau(A)=\frac{4}{3}$ *and the Zariski Decomposition of the divisor* $-K_S-vA$ *is given by:*

a). $$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}B \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{2}B-(v-1)(B_1+B_2+B_3+B_4+B_5+B_6) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{2}B \text{ if } v\in[0,1],\\ \frac{v}{2}B+(v-1)(B_1+B_2+B_3+B_4+B_5+B_6) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$

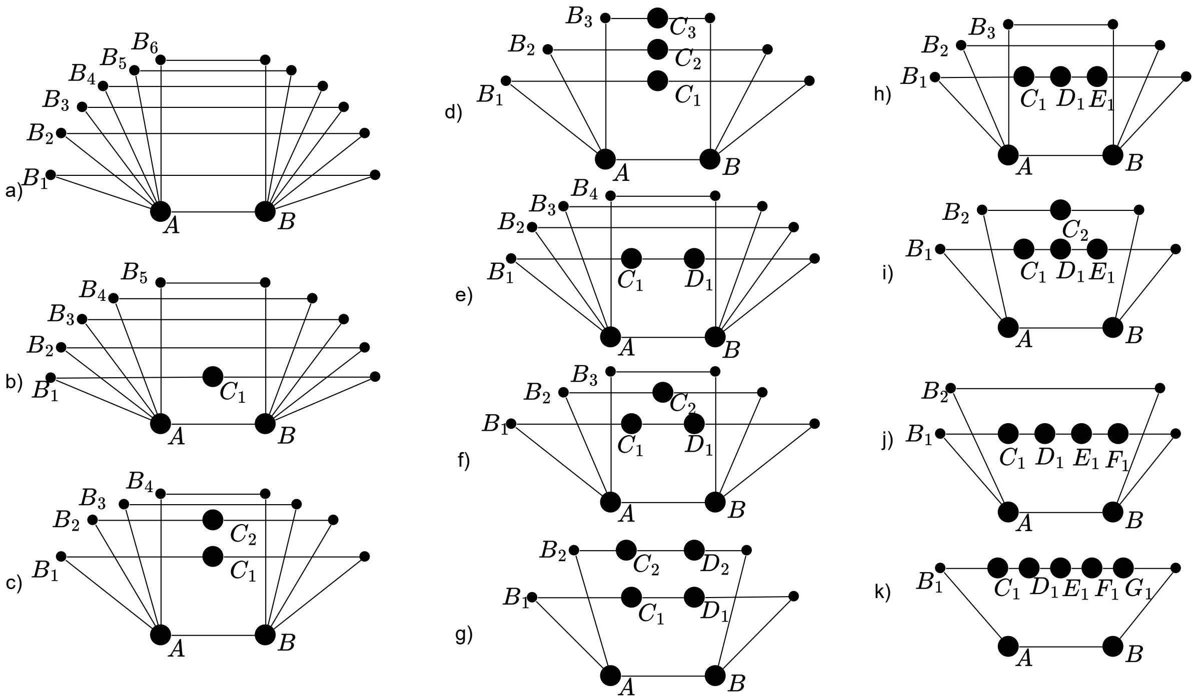


Figure 8.12: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{9}{7}$

b).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B - (v-1)(2B_1 + C_1 + B_2 + B_3 + B_4 + B_5) \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{2}B \text{ if } v \in [0,1], \\ \frac{v}{2}B + (v-1)(2B_1 + C_1 + B_2 + B_3 + B_4 + B_5) \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

c).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B - (v-1)(2B_1 + C_1 + 2B_2 + C_2 + B_3 + B_4) \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{2}B \text{ if } v \in [0,1], \\ \frac{v}{2}B + (v-1)(2B_1 + C_1 + 2B_2 + C_2 + B_3 + B_4) \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

d).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B - (v-1)(2B_1 + C_1 + 2B_2 + C_2 + 2B_3 + C_3) \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{2}B \text{ if } v \in [0,1], \\ \frac{v}{2}B + (v-1)(2B_1 + C_1 + 2B_2 + C_2 + 2B_3 + C_3) \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

e).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}B \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}B - (v-1)(3B_1 + 2C_1 + D_1 + B_2 + B_3 + B_4) \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{2}B \text{ if } v \in [0,1], \\ \frac{v}{2}B + (v-1)(3B_1 + 2C_1 + D_1 + B_2 + B_3 + B_4) \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

f).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}B \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{2}B-(v-1)(3B_1+2C_1+D_1+2B_2+C_2+B_3) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{2}B \text{ if } v\in[0,1],\\ \frac{v}{2}B+(v-1)(3B_1+2C_1+D_1+2B_2+C_2+B_3) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$

g).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}B \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{2}B-(v-1)(3B_1+2C_1+D_1+3B_2+2C_2+D_2) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{2}B \text{ if } v\in[0,1],\\ \frac{v}{2}B+(v-1)(3B_1+2C_1+D_1+3B_2+2C_2+D_2) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$

h).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}B \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{2}B-(v-1)(4B_1+3C_1+2D_1+E_1+B_2+B_3) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{2}B \text{ if } v\in[0,1],\\ \frac{v}{2}B+(v-1)(4B_1+3C_1+2D_1+E_1+B_2+B_3) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$

i).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}B \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{2}B-(v-1)(4B_1+3C_1+2D_1+E_1+2B_2+C_2) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{2}B \text{ if } v\in[0,1],\\ \frac{v}{2}B+(v-1)(4B_1+3C_1+2D_1+E_1+2B_2+C_2) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$

j).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}B \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{2}B-(v-1)(5B_1+4C_1+3D_1+2E_1+F_1+B_2) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{2}B \text{ if } v\in[0,1],\\ \frac{v}{2}B+(v-1)(5B_1+4C_1+3D_1+2E_1+F_1+B_2) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$

k).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}B \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{2}B-(v-1)(6B_1+5C_1+4D_1+3E_1+2F_1+G_1) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{2}B \text{ if } v\in[0,1],\\ \frac{v}{2}B+(v-1)(6B_1+5C_1+4D_1+3E_1+2F_1+G_1) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}2-\frac{3v^2}{2} \text{ if } v\in[0,1],\\ \frac{(4-3v)^2}{2} \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}\quad P(v)\cdot A=\begin{cases}\frac{3v}{2} \text{ if } v\in[0,1],\\ 3(2-\frac{3v}{2}) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$

In this case $\delta_P(S)=\frac{9}{7}$ *if* $P\in A\backslash B$.

Proof. The Zariski Decomposition in part a). follows from

$$-K_S - vA \sim_{\mathbb{R}} \Big(\frac{4}{3} - v\Big)A + \frac{1}{3}\Big(2B + B_1 + B_2 + B_3 + B_4 + B_5 + B_6\Big).$$

A similar statement holds in other parts. We have $S_S(A) = \frac{7}{9}$. Thus, $\delta_P(S) \leq \frac{9}{7}$ for $P \in A$. Note that for $P \in A \backslash B$ we have:

$$h(v) \leq \begin{cases} \frac{9v^2}{8} \text{ if } v \in [0,1], \\ \frac{9(4-3v)(5v-4)}{8} \text{ if } v \in \big[1, \frac{4}{3}\big]. \end{cases}$$

So we have $S(W^A_{\bullet,\bullet}; P) \leq \frac{2}{3} < \frac{7}{9}$. Thus, $\delta_P(S) = \frac{9}{7}$ if $P \in A \backslash B$. □

Lemma 8.1.15. *Suppose $P = A_1 \cap A_2$ where A_1 and A_2 are (-2)-curves disjoint from other (-2)-curves, and B is a unique (0)-curve containing P. Consider the blowup $\sigma : \widetilde{S} \to S$ of S at P with the exceptional divisor E_P.*

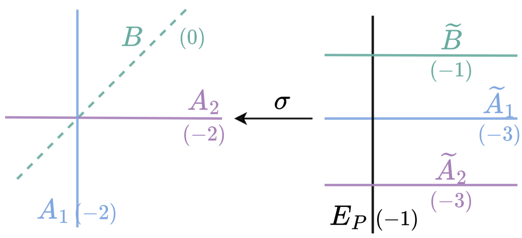


Figure 8.13: Picture: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{6}{5}$ (intersection of two (-2)-curves)

Then $\tau(A) = 3$ and the Zariski decomposition of the divisor $\sigma^*(-K_S) - vE_P$ is given by:

$$P(v) = \begin{cases} \sigma^*(-K_S) - vE_P - \frac{v}{3}(\widetilde{A}_1 + \widetilde{A}_2) \text{ if } v \in [0,2], \\ \sigma^*(-K_S) - vE_P - \frac{v}{3}(\widetilde{A}_1 + \widetilde{A}_2) - (v-2)\widetilde{B} \text{ if } v \in [2,3]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(\widetilde{A}_1 + \widetilde{A}_2) \text{ if } v \in [0,2], \\ \frac{v}{3}(\widetilde{A}_1 + \widetilde{A}_2) + (v-2)\widetilde{B} \text{ if } v \in [2,3]. \end{cases}$$

Moreover,

$$P(v)^2 = \begin{cases} 2 - \frac{v^2}{3} \text{ if } v \in [0,2], \\ \frac{2(3-v)^2}{3} \text{ if } v \in [2,3]. \end{cases} \quad P(v) \cdot E_P = \begin{cases} \frac{v}{3} \text{ if } v \in [0,2], \\ 2(1 - \frac{v}{3}) \text{ if } v \in [2,3]. \end{cases}$$

In this case $\delta_P(S) = \frac{6}{5}$ for $P = A_1 \cap A_2$.

Proof. The Zariski Decomposition follows from $\sigma^*(-K_S)-vE_P\sim_{\mathbb{R}}(3-v)E_P+\widetilde{A}_1+\widetilde{A}_2+\widetilde{B}$ where $\widetilde{A}_1$, $\widetilde{A}_1$ and $\widetilde{B}$ is are strict transforms of A_1, A_2 and B respectively and E_P is the exceptional divisor. We have $S_S(E_P)=\frac{5}{3}$. Thus, $\delta_P(S)\le\frac{2}{5/3}=\frac{6}{5}$. Moreover if $O\in E_P\backslash(\widetilde{A}_1\cup\widetilde{A}_2)$ or if $O\in E_P\cap(\widetilde{A}_1\cup\widetilde{A}_2)$:

$$h(v)\le\begin{cases}\frac{v^2}{18}\text{ if }v\in[0,2],\\ \frac{2(3-v)(2v-3)}{9}\text{ if }v\in[2,3].\end{cases}\quad\text{or }h(v)=\begin{cases}\frac{v^2}{6}\text{ if }v\in[0,2],\\ \frac{(3-v)(v+6)}{9}\text{ if }v\in[2,3].\end{cases}$$

Thus, $S(W^{E_P}_{\bullet,\bullet};O)=\frac{1}{3}\le\frac{5}{6}$ or $S(W^{E_P}_{\bullet,\bullet};O)=\frac{7}{9}\le\frac{5}{6}$. We get that $\delta_P(S)=\frac{6}{5}$ for $P=A_1\cap A_2$. $\square$

Lemma 8.1.16. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

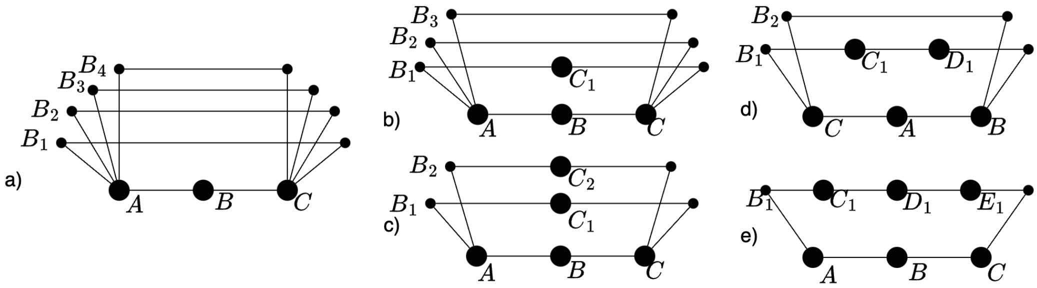


Figure 8.14: Dual graph: $(-K_S)^2=2$ and $\delta_P(S)=\frac{6}{5}$

Then $\tau(A)=\frac{3}{2}$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

a).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{3}(2B+C)\ \textit{if } v\in[0,1],\\ -K_S-vA-\frac{v}{3}(2B+C)-(v-1)(B_1+B_2+B_3+B_4)\ \textit{if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{3}(2B+C)\ \textit{if } v\in[0,1],\\ \frac{v}{3}(2B+C)+(v-1)(B_1+B_2+B_3+B_4)\ \textit{if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

b).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{3}(2B+C)\ \textit{if } v\in[0,1],\\ -K_S-vA-\frac{v}{3}(2B+C)-(v-1)(2B_1+C_1+B_2+B_3)\ \textit{if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{3}(2B+C)\ \textit{if } v\in[0,1],\\ \frac{v}{3}(2B+C)+(v-1)(2B_1+C_1+B_2+B_3)\ \textit{if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

c).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{3}(2B+C)\ \textit{if } v\in[0,1],\\ -K_S-vA-\frac{v}{3}(2B+C)-(v-1)(2B_1+C_1+2B_2+C_2)\ \textit{if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{3}(2B+C)\ \textit{if } v\in[0,1],\\ \frac{v}{3}(2B+C)+(v-1)(2B_1+C_1+2B_2+C_2)\ \textit{if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

d). $$P(v)=\begin{cases}-K_S-vA-\frac{v}{3}(2B+C) \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{3}(2B+C)-(v-1)(3B_1+2C_1+D_1+B_2) \text{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{3}(2B+C) \text{ if } v\in[0,1],\\ \frac{v}{3}(2B+C)+(v-1)(3B_1+2C_1+D_1+B_2) \text{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

e). $$P(v)=\begin{cases}-K_S-vA-\frac{v}{3}(2B+C) \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{3}(2B+C)-(v-1)(4B_1+3C_1+2D_1+E_1) \text{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{3}(2B+C) \text{ if } v\in[0,1],\\ \frac{v}{3}(2B+C)+(v-1)(4B_1+3C_1+2D_1+E_1) \text{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}2-\frac{4v^2}{3} \text{ if } v\in[0,1],\\ \frac{2(3-2v)^2}{3} \text{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}\qquad P(v)\cdot A=\begin{cases}\frac{4v}{3} \text{ if } v\in[0,1],\\ 4(1-\frac{2v}{3}) \text{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

In this case $\delta_P(S)=\frac{6}{5}$ if $P\in A\backslash B$.

Proof. The Zariski Decomposition in part a). follows from

$$-K_S-vA\sim_{\mathbb{R}}\Big(\frac{3}{2}-v\Big)A+\frac{1}{2}\Big(2B+C+B_1+B_2+B_3+B_4\Big).$$

A similar statement holds in other parts. We have $S_S(A)=\frac{5}{6}$. Thus, $\delta_P(S)\leq\frac{6}{5}$ for $P\in A$. Note that for $P\in A\backslash B$ we have:

$$h(v)\leq\begin{cases}\frac{8v^2}{9} \text{ if } v\in[0,1],\\ \frac{4(3-2v)(3-v)}{9} \text{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

So we have $S(W^A_{\bullet,\bullet};P)\leq\frac{1}{2}<\frac{5}{6}$. Thus, $\delta_P(S)=\frac{6}{5}$ if $P\in A\backslash B$. □

Lemma 8.1.17. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

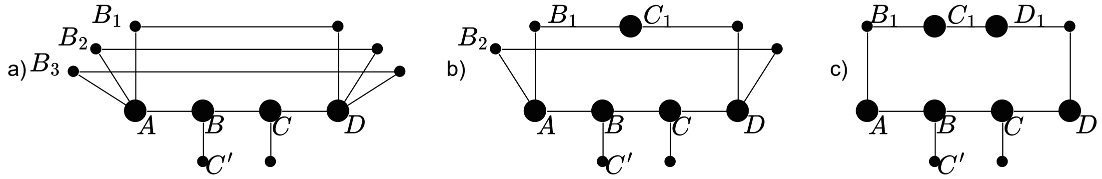


Figure 8.15: Dual graph: $(-K_S)^2=2$ and $\delta_P(S)=\frac{36}{31}$

Then $\tau(A)=\frac{3}{2}$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

a).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{4}(3B+2C+D) \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{4}(3B+2C+D)-(v-1)(B_1+B_2+B_3) \text{ if } v\in\big[1,\frac{4}{3}\big],\\ -K_S-vA-(v-1)(3B+2C+D+B_1+B_2+B_3)-(3v-4)C' \text{ if } v\in\big[\frac{4}{3},\frac{3}{2}\big].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{4}(3B+2C+D) \text{ if } v\in[0,1],\\ \frac{v}{4}(3B+2C+D)+(v-1)(B_1+B_2+B_3) \text{ if } v\in\big[1,\frac{4}{3}\big],\\ (v-1)(3B+2C+D+B_1+B_2+B_3)+(3v-4)C' \text{ if } v\in\big[\frac{4}{3},\frac{3}{2}\big].\end{cases}$$

b).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{4}(3B+2C+D) \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{4}(3B+2C+D)-(v-1)(2B_1+C_1+B_2) \text{ if } v\in\big[1,\frac{4}{3}\big],\\ -K_S-vA-(v-1)(3B+2C+D+2B_1+C_1+B_2)-(3v-4)C' \text{ if } v\in\big[\frac{4}{3},\frac{3}{2}\big].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{4}(3B+2C+D) \text{ if } v\in[0,1],\\ \frac{v}{4}(3B+2C+D)+(v-1)(2B_1+C_1+B_2) \text{ if } v\in\big[1,\frac{4}{3}\big],\\ (v-1)(3B+2C+D+2B_1+C_1+B_2)+(3v-4)C' \text{ if } v\in\big[\frac{4}{3},\frac{3}{2}\big].\end{cases}$$

c).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{4}(3B+2C+D) \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{4}(3B+2C+D)-(v-1)(3B_1+2C_1+D_1) \text{ if } v\in\big[1,\frac{4}{3}\big],\\ -K_S-vA-(v-1)(3B+2C+D+3B_1+2C_1+D_1)-(3v-4)C' \text{ if } v\in\big[\frac{4}{3},\frac{3}{2}\big].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{4}(3B+2C+D) \text{ if } v\in[0,1],\\ \frac{v}{4}(3B+2C+D)+(v-1)(3B_1+2C_1+D_1) \text{ if } v\in\big[1,\frac{4}{3}\big],\\ (v-1)(3B+2C+D+3B_1+2C_1+D_1)+(3v-4)C' \text{ if } v\in\big[\frac{4}{3},\frac{3}{2}\big].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}2-\frac{5v^2}{4} \text{ if } v\in[0,1],\\ \frac{(10-7v)(2-v)}{4} \text{ if } v\in\big[1,\frac{4}{3}\big],\\ (3-2v)^2 \text{ if } v\in\big[\frac{4}{3},\frac{3}{2}\big].\end{cases}\quad P(v)\cdot A=\begin{cases}\frac{5v}{4} \text{ if } v\in[0,1],\\ 3-\frac{7v}{4} \text{ if } v\in\big[1,\frac{4}{3}\big],\\ 2(3-2v) \text{ if } v\in\big[\frac{4}{3},\frac{3}{2}\big].\end{cases}$$

In this case $\delta_P(S)=\frac{36}{31}$ if $P\in A\backslash B$.

Proof. In part a). the Zariski Decomposition follows from

$$-K_S-vA\sim_{\mathbb{R}}\Big(\frac{3}{2}-v\Big)A+\frac{1}{2}\Big(3B+2C+D+B_1+B_2+B_3+C'\Big).$$

A similar statement holds in other parts. We have $S_S(A) = \frac{31}{36}$. Thus, $\delta_P(S) \le \frac{36}{31}$ for $P \in A$. Note that for $P \in A \backslash B$ we have:

$$h(v) \le \begin{cases} \frac{25v^2}{32} \text{ if } v \in [0,1], \\ \frac{(12-7v)(17v-12)}{32} \text{ if } v \in \left[1, \frac{4}{3}\right], \\ 2(3-2v)v \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right] \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \le \frac{26}{36} < \frac{31}{36}$. Thus, $\delta_P(S) = \frac{36}{31}$ if $P \in A \backslash B$. □

Lemma 8.1.18. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

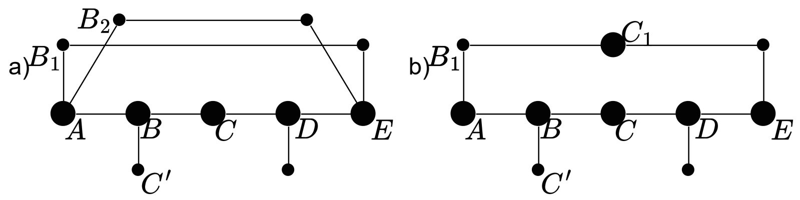


Figure 8.16: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{8}{7}$

Then $\tau(A) = \frac{3}{2}$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{5}(4B+3C+2D+E) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{5}(4B+3C+2D+E) - (v-1)(B_1+B_2) \text{ if } v \in \left[1, \frac{5}{4}\right], \\ -K_S - vA - (v-1)(4B+3C+2D+E+B_1+B_2) - (4v-5)C' \text{ if } v \in \left[\frac{5}{4}, \frac{3}{2}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(4B+3C+2D+E) \text{ if } v \in [0,1], \\ \frac{v}{5}(4B+3C+2D+E) + (v-1)(B_1+B_2) \text{ if } v \in \left[1, \frac{5}{4}\right], \\ (v-1)(4B+3C+2D+E+B_1+B_2) + (4v-5)C' \text{ if } v \in \left[\frac{5}{4}, \frac{3}{2}\right]. \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{5}(4B+3C+2D+E) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{5}(4B+3C+2D+E) - (v-1)(2B_1+C_1) \text{ if } v \in \left[1, \frac{5}{4}\right], \\ -K_S - vA - (v-1)(4B+3C+2D+E+2B_1+C_1) - (4v-5)C' \text{ if } v \in \left[\frac{5}{4}, \frac{3}{2}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(4B+3C+2D+E) \text{ if } v \in [0,1], \\ \frac{v}{5}(4B+3C+2D+E) + (v-1)(2B_1+C_1) \text{ if } v \in \left[1, \frac{5}{4}\right], \\ (v-1)(4B+3C+2D+E+2B_1+C_1) + (4v-5)C' \text{ if } v \in \left[\frac{5}{4}, \frac{3}{2}\right]. \end{cases}$$

$$(P(v))^2 = \begin{cases} 2 - \frac{6v^2}{5} \text{ if } v \in [0,1], \\ 4 - 4v + \frac{4v^2}{5} \text{ if } v \in \left[1, \frac{5}{4}\right], \\ (3-2v)^2 \text{ if } v \in \left[\frac{5}{4}, \frac{3}{2}\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{6v}{5} \text{ if } v \in [0,1], \\ 2(1 - \frac{2v}{5}) \text{ if } v \in \left[1, \frac{5}{4}\right], \\ 2(3-2v) \text{ if } v \in \left[\frac{5}{4}, \frac{3}{2}\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{8}{7}$ *if* $P \in A \backslash B$.

Proof. In part a). the Zariski Decomposition follows from

$$-K_S - vA \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)A + \frac{1}{2}\Big(4B + 3C + 2D + E + B_1 + B_2 + 2C'\Big).$$

A similar statement holds in other parts. We have $S_S(A) = \frac{7}{8}$. Thus, $\delta_P(S) \leq \frac{8}{7}$ for $P \in A$. Note that for $P \in A \backslash B$ we have:

$$h(v) \leq \begin{cases} \frac{18v^2}{25} \text{ if } v \in [0,1], \\ \frac{2(5-2v)(8v-5)}{25} \text{ if } v \in \big[1, \frac{5}{4}\big], \\ 2(3-2v) \text{ if } v \in \big[\frac{5}{4}, \frac{3}{2}\big]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \leq \frac{7}{12} < \frac{7}{8}$. Thus, $\delta_P(S) = \frac{8}{7}$ if $P \in A \backslash B$. □

Lemma 8.1.19. *Suppose* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

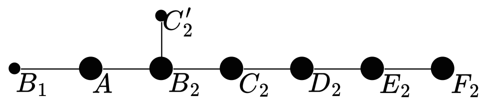


Figure 8.17: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{60}{53}$

Then $\tau(A) = \frac{3}{2}$ *and the Zariski Decomposition of the divisor* $-K_S - vA$ *is given by:*

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{6}(5B_2 + 4C_2 + 3D_2 + 2E_2 + F_2) \textit{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{6}(5B_2 + 4C_2 + 3D_2 + 2E_2 + F_2) - (v-1)B_1 \textit{ if } v \in \big[1, \frac{6}{5}\big], \\ -K_S - vA - (v-1)(5B_2 + 4C_2 + 3D_2 + 2E_2 + F_2 + B_1) - (5v-6)C_2' \textit{ if } v \in \big[\frac{6}{5}, \frac{3}{2}\big]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(5B_2 + 4C_2 + 3D_2 + 2E_2 + F_2) \textit{ if } v \in [0,1], \\ \frac{v}{6}(5B_2 + 4C_2 + 3D_2 + 2E_2 + F_2) + (v-1)B_1 \textit{ if } v \in \big[1, \frac{6}{5}\big], \\ (v-1)(5B_2 + 4C_2 + 3D_2 + 2E_2 + F_2 + B_1) + (5v-6)C_2' \textit{ if } v \in \big[\frac{6}{5}, \frac{3}{2}\big]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2 - \frac{7v^2}{6} \textit{ if } v \in [0,1], \\ 3 - 2v - \frac{v^2}{6} \textit{ if } v \in \big[1, \frac{6}{5}\big], \\ (3-2v)^2 \textit{ if } v \in \big[\frac{6}{5}, \frac{3}{2}\big]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{7v}{6} \textit{ if } v \in [0,1], \\ 1 + \frac{v}{6} \textit{ if } v \in \big[1, \frac{6}{5}\big], \\ 2(3-2v) \textit{ if } v \in \big[\frac{6}{5}, \frac{3}{2}\big]. \end{cases}$$

In this case $\delta_P(S) = \frac{60}{53}$ *if* $P \in A \backslash B_2$.

Proof. The Zariski Decomposition follows from

$$-K_S - vA \sim_{\mathbb{R}} \Big(\frac{3}{2}-v\Big)A + \frac{1}{2}\Big(5B_2+4C_2+3D_2+2E_2+F_2+B_1+3C_2'\Big).$$

We have $S_S(A)=\frac{53}{60}$. Thus, $\delta_P(S)\le\frac{60}{53}$ for $P\in A$. Note that for $P\in A\backslash B_2$ we have:

$$h(v)\le\begin{cases}\frac{49v^2}{72} \text{ if } v\in[0,1],\\ \frac{(v+6)(13v-6)}{72} \text{ if } v\in\big[1,\frac{6}{5}\big],\\ 2(3-2v)(2-v) \text{ if } v\in\big[\frac{6}{5},\frac{3}{2}\big].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\le\frac{31}{60}<\frac{53}{60}$. Thus, $\delta_P(S)=\frac{60}{53}$ if $P\in A\backslash B_2$. □

Lemma 8.1.20. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

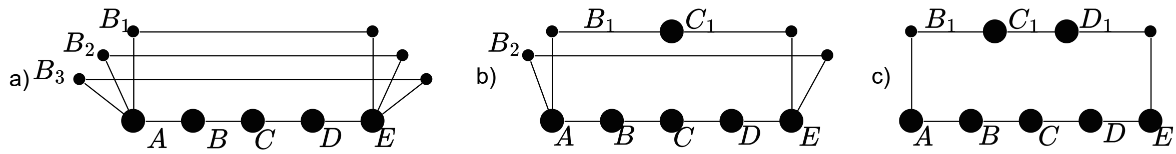


Figure 8.18: Dual graph: $(-K_S)^2=2$ and $\delta_P(S)=\frac{9}{8}$ with $\tau(A)=\frac{5}{3}$

Then $\tau(A)=\frac{5}{3}$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

a).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{5}(4B+3C+2D+E) \textit{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{5}(4B+3C+2D+E)-(v-1)(B_1+B_2+B_3) \textit{ if } v\in\big[1,\frac{5}{3}\big].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{5}(4B+3C+2D+E) \textit{ if } v\in[0,1],\\ \frac{v}{5}(4B+3C+2D+E)+(v-1)(B_1+B_2+B_3) \textit{ if } v\in\big[1,\frac{5}{3}\big].\end{cases}$$

b).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{5}(4B+3C+2D+E) \textit{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{5}(4B+3C+2D+E)-(v-1)(2B_1+C_1+B_2) \textit{ if } v\in\big[1,\frac{5}{3}\big].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{5}(4B+3C+2D+E) \textit{ if } v\in[0,1],\\ \frac{v}{5}(4B+3C+2D+E)+(v-1)(2B_1+C_1+B_2) \textit{ if } v\in\big[1,\frac{5}{3}\big].\end{cases}$$

c).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{5}(4B+3C+2D+E) \textit{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{5}(4B+3C+2D+E)-(v-1)(3B_1+2C_1+D_1) \textit{ if } v\in\big[1,\frac{5}{3}\big].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{5}(4B+3C+2D+E) \textit{ if } v\in[0,1],\\ \frac{v}{5}(4B+3C+2D+E)+(v-1)(3B_1+2C_1+D_1) \textit{ if } v\in\big[1,\frac{5}{3}\big].\end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2-\frac{6v^2}{5} \text{ if } v\in[0,1], \\ \frac{(5-3v)^2}{5} \text{ if } v\in\left[1,\frac{5}{3}\right]. \end{cases} \quad P(v)\cdot A = \begin{cases} \frac{6v}{5} \text{ if } v\in[0,1], \\ 3(1-\frac{3v}{5}) \text{ if } v\in\left[1,\frac{5}{3}\right]. \end{cases}$$

In this case $\delta_P(S)=\frac{9}{8}$ *if* $P\in A\backslash B$.

Proof. In part a). the Zariski Decomposition follows from

$$-K_S - vA \sim_{\mathbb{R}} \Big(\frac{5}{3}-v\Big)A + \frac{1}{3}\Big(4B+3C+2D+E+2B_1+2B_2+2B_3\Big).$$

A similar statement holds in other parts. We have $S_S(A)=\frac{8}{9}$. Thus, $\delta_P(S)\le\frac{9}{8}$ for $P\in A$. Note that for $P\in A\backslash B$ we have:

$$h(v)\le \begin{cases} \frac{18v^2}{25} \text{ if } v\in[0,1], \\ \frac{9(5-3v)(7v-5)}{50} \text{ if } v\in\left[1,\frac{5}{3}\right]. \end{cases}$$

So we have $S(W^A_{\bullet,\bullet};P)\le\frac{2}{3}<\frac{8}{9}$. Thus, $\delta_P(S)=\frac{9}{8}$ if $P\in A\backslash B$. □

Lemma 8.1.21. *Suppose* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

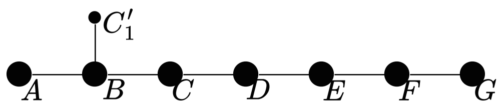


Figure 8.19: Dual graph: $(-K_S)^2=2$ and $\delta_P(S)=\frac{9}{8}$ with $\tau(A)=\frac{3}{2}$

Then $\tau(A)=\frac{3}{2}$ *and the Zariski Decomposition of the divisor* $-K_S-vA$ *is given by:*

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{7}(6B+5C+4D+3E+2F+G) \text{ if } v\in\left[0,\frac{7}{6}\right], \\ -K_S - vA - (v-1)(6B+5C+4D+3E+2F+G) - (6v-7)C' \text{ if } v\in\left[\frac{7}{6},\frac{3}{2}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{7}(6B+5C+4D+3E+2F+G) \text{ if } v\in\left[0,\frac{7}{6}\right], \\ (v-1)(6B+5C+4D+3E+2F+G) + (6v-7)C' \text{ if } v\in\left[\frac{7}{6},\frac{3}{2}\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2-\frac{8v^2}{7} \text{ if } v\in\left[0,\frac{7}{6}\right], \\ (3-2v)^2 \text{ if } v\in\left[\frac{7}{6},\frac{3}{2}\right]. \end{cases} \quad P(v)\cdot A = \begin{cases} \frac{8v}{7} \text{ if } v\in\left[0,\frac{7}{6}\right], \\ 2(3-2v) \text{ if } v\in\left[\frac{7}{6},\frac{3}{2}\right]. \end{cases}$$

In this case $\delta_P(S)=\frac{9}{8}$ *if* $P\in A\backslash B$.

Proof. The Zariski Decomposition follows from

$$-K_S - vA \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)A + \frac{1}{2}\Big(6B + 5C + 4D + 3E + 2F + G + 4C'\Big).$$

We have $S_S(A) = \frac{8}{9}$. Thus, $\delta_P(S) \le \frac{9}{8}$ for $P \in A$. Note that for $P \in A \backslash B$ we have:

$$h(v) \le \begin{cases} \frac{32v^2}{49} \text{ if } v \in \big[0, \frac{7}{6}\big], \\ 2(3-2v)^2 \text{ if } v \in \big[\frac{7}{6}, \frac{3}{2}\big]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \le \frac{4}{9} < \frac{8}{9}$. Thus, $\delta_P(S) = \frac{9}{8}$ if $P \in A \backslash B$. □

Lemma 8.1.22. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

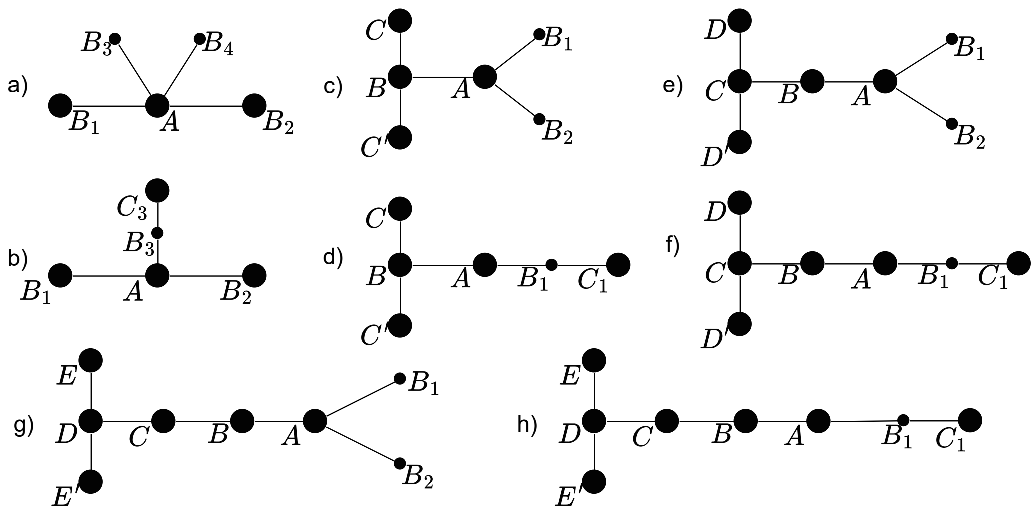


Figure 8.20: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = 1$

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}(B_1 + B_2) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}(B_1 + B_2) - (v-1)(B_3 + B_4) \text{ if } v \in [1,2]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(B_1 + B_2) \text{ if } v \in [0,1], \\ \frac{v}{2}(B_1 + B_2) + (v-1)(B_3 + B_4) \text{ if } v \in [1,2]. \end{cases}$$

b).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{2}(B_1 + B_2) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{2}(B_1 + B_2) - (v-1)(2B_3 + C_3) \text{ if } v \in [1,2]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(B_1 + B_2) \text{ if } v \in [0,1], \\ \frac{v}{2}(B_1 + B_2) + (v-1)(2B_3 + C_3) \text{ if } v \in [1,2]. \end{cases}$$

c).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}(2B+C+C') \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{2}(2B+C+C')-(v-1)(B_1+B_2) \text{ if } v\in[1,2].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{2}(2B+C+C') \text{ if } v\in[0,1],\\ \frac{v}{2}(2B+C+C')+(v-1)(B_1+B_2) \text{ if } v\in[1,2].\end{cases}$$

d).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}(2B+C+C') \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{2}(2B+C+C')-(v-1)(2B_1+C_1) \text{ if } v\in[1,2].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{2}(2B+C+C') \text{ if } v\in[0,1],\\ \frac{v}{2}(2B+C+C')+(v-1)(2B_1+C_1) \text{ if } v\in[1,2].\end{cases}$$

e).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}(2B+2C+D+D') \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{2}(2B+2C+D+D')-(v-1)(B_1+B_2) \text{ if } v\in[1,2].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{2}(2B+2C+D+D') \text{ if } v\in[0,1],\\ \frac{v}{2}(2B+2C+D+D')+(v-1)(B_1+B_2) \text{ if } v\in[1,2].\end{cases}$$

f).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}(2B+2C+D+D') \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{2}(2B+2C+D+D')-(v-1)(2B_1+C_1) \text{ if } v\in[1,2].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{2}(2B+2C+D+D') \text{ if } v\in[0,1],\\ \frac{v}{2}(2B+2C+D+D')+(v-1)(2B_1+C_1) \text{ if } v\in[1,2].\end{cases}$$

g).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}(2B+2C+2D+E+E') \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{2}(2B+2C+2D+E+E')-(v-1)(B_1+B_2) \text{ if } v\in[1,2].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{2}(2B+2C+2D+E+E') \text{ if } v\in[0,1],\\ \frac{v}{2}(2B+2C+2D+E+E')+(v-1)(B_1+B_2) \text{ if } v\in[1,2].\end{cases}$$

h).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{2}(2B+2C+2D+E+E') \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{2}(2B+2C+2D+E+E')-(v-1)(2B_1+C_1) \text{ if } v\in[1,2].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{2}(2B+2C+2D+E+E') \text{ if } v\in[0,1],\\ \frac{v}{2}(2B+2C+2D+E+E')+(v-1)(2B_1+C_1) \text{ if } v\in[1,2].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}2-v^2 \text{ if } v\in[0,1],\\ (2-v)^2 \text{ if } v\in[1,2].\end{cases}\qquad P(v)\cdot A=\begin{cases}v \text{ if } v\in[0,1],\\ 2-v \text{ if } v\in[1,2].\end{cases}$$

In this case $\delta_P(S)=1$ *if* $P\in A\backslash B$.

Proof. The Zariski Decomposition in part a). follows from $-K_S - vA \sim_{\mathbb{R}} \big(2-v\big)A + B_1 + B_2 + B_3 + B_4$. A similar statement holds in other parts. We have $S_S(A) = 1$. Thus, $\delta_P(S) \le 1$ for $P \in A$. Note that for $P \in A \backslash (B_1 \cup B_2 \cup B)$ or $P \in A \cap (B_1 \cup B_2)$ we have:

$$h(v) \le \begin{cases} v^2 \text{ if } v \in [0,1], \\ 2-v \text{ if } v \in [1,2]. \end{cases} \quad \text{or } h(v) \le \begin{cases} \frac{v^2}{2} \text{ if } v \in [0,1], \\ \frac{(2-v)v}{2} \text{ if } v \in [1,2]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \le \frac{5}{6} < 1$ or $S(W^A_{\bullet,\bullet};P) \le \frac{1}{2} < 1$. Thus, $\delta_P(S) = 1$ if $P \in A$. □

Lemma 8.1.23. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

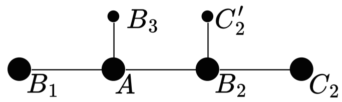


Figure 8.21: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{12}{13}$

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{6}(3B_1 + 4B_2 + 2C_2) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{6}(3B_1 + 4B_2 + 2C_2) - (v-1)B_3 \text{ if } v \in \big[1, \frac{3}{2}\big], \\ -K_S - vA - (v-1)(2B_2 + C_2 + B_3) - \frac{v}{2}B_1 - (2v-3)C_2' \text{ if } v \in \big[\frac{3}{2}, 2\big]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(3B_1 + 4B_2 + 2C_2) \text{ if } v \in [0,1], \\ \frac{v}{6}(3B_1 + 4B_2 + 2C_2) + (v-1)B_3 \text{ if } v \in \big[1, \frac{3}{2}\big], \\ (v-1)(2B_2 + C_2 + B_3) + \frac{v}{2}B_1 + (2v-3)C_2' \text{ if } v \in \big[\frac{3}{2}, 2\big]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2 - \frac{5v^2}{6} \text{ if } v \in [0,1], \\ 3 - 2v + \frac{v^2}{6} \text{ if } v \in \big[1, \frac{3}{2}\big], \\ \frac{3(2-v)^2}{2} \text{ if } v \in \big[\frac{3}{2}, 2\big]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{5v}{6} \text{ if } v \in [0,1], \\ 1 - \frac{v}{6} \text{ if } v \in \big[1, \frac{3}{2}\big], \\ 3(1 - \frac{v}{2}) \text{ if } v \in \big[\frac{3}{2}, 2\big]. \end{cases}$$

In this case $\delta_P(S) = \frac{12}{13}$ if $P \in A$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + B_1 + 2B_2 + C_2 + C_2' + B_3$. We have $S_S(A) = \frac{13}{12}$. Thus, $\delta_P(S) \le \frac{12}{13}$ for $P \in A$. Note that if $P \in A \cap B_2$ or if $P \in A \backslash B_2$ we have:

$$h(v) \le \begin{cases} \frac{65v^2}{72} \text{ if } v \in [0,1], \\ \frac{(6-v)(7v+6)}{72} \text{ if } v \in \left[1, \frac{3}{2}\right], \\ \frac{3(2-v)(5v-2)}{8} \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases} \quad \text{or } h(v) = \begin{cases} \frac{25v^2}{72} \text{ if } v \in [0,1], \\ \frac{(6-v)(11v-6)}{72} \text{ if } v \in \left[1, \frac{3}{2}\right], \\ \frac{3(2-v)(v+2)}{8} \text{ if } v \in \left[\frac{3}{2}, 2\right] \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) = \frac{13}{12}$ or $S(W^A_{\bullet,\bullet};P) = \frac{13}{24} < \frac{13}{12}$. Thus, $\delta_P(S) = \frac{12}{13}$ if $P \in A$. □

Lemma 8.1.24. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

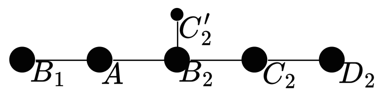


Figure 8.22: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{9}{10}$ with $-K_S - vA$ nef on $\left[0, \frac{4}{3}\right]$

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{4}(2B_1 + 3B_2 + 2C_2 + D_2) \text{ if } v \in \left[0, \frac{4}{3}\right], \\ -K_S - vA - \frac{v}{2}B_1 - (v-1)(3B_2 + 2C_2 + D_2) - (3v-4)C_2' \text{ if } v \in \left[\frac{4}{3}, 2\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(2B_1 + 3B_2 + 2C_2 + D_2) \text{ if } v \in \left[0, \frac{4}{3}\right], \\ \frac{v}{2}B_1 + (v-1)(3B_2 + 2C_2 + D_2) + (3v-4)C_2' \text{ if } v \in \left[\frac{4}{3}, 2\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2 - \frac{3v^2}{4} \text{ if } v \in \left[0, \frac{4}{3}\right], \\ \frac{3(2-v)^2}{2} \text{ if } v \in \left[\frac{4}{3}, 2\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{3v}{4} \text{ if } v \in \left[0, \frac{4}{3}\right], \\ 3(1 - \frac{v}{2}) \text{ if } v \in \left[\frac{4}{3}, 2\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{9}{10}$ if $P \in A \backslash B_2$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + B_1 + 3B_2 + 2C_2 + D_2 + 2C_2'$. We have $S_S(A) = \frac{10}{9}$. Thus, $\delta_P(S) \le \frac{9}{10}$ for $P \in A$. Note that for $P \in A \backslash B_2$ we have:

$$h(v) \le \begin{cases} \frac{21v^2}{32} \text{ if } v \in \left[0, \frac{4}{3}\right], \\ \frac{3(2-v)(6-v)}{8} \text{ if } v \in \left[\frac{4}{3}, 2\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \le \frac{8}{9} < \frac{10}{9}$. Thus, $\delta_P(S) = \frac{9}{10}$ if $P \in A \backslash B_2$. □

Lemma 8.1.25. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

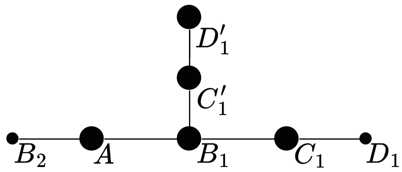


Figure 8.23: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{9}{10}$ with $-K_S - vA$ nef on $[0,1]$

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{5}(3C_1 + 6B_1 + 4C_1' + 2D_1') \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{5}(3C_1 + 6B_1 + 4C_1' + 2D_1') - (v-1)B_2 \text{ if } v \in \left[1, \frac{5}{3}\right], \\ -K_S - vA - (v-1)(3B_1 + 2C_1' + D_1' + B_2) - (3v-4)C_1 - (3v-5)D_1 \text{ if } v \in \left[\frac{5}{3}, 2\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(3C_1 + 6B_1 + 4C_1' + 2D_1') \text{ if } v \in [0,1], \\ \frac{v}{5}(3C_1 + 6B_1 + 4C_1' + 2D_1') + (v-1)B_2 \text{ if } v \in \left[1, \frac{5}{3}\right], \\ (v-1)(3B_1 + 2C_1' + D_1' + B_2) + (3v-4)C_1 + (3v-5)D_1 \text{ if } v \in \left[\frac{5}{3}, 2\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2 - \frac{4v^2}{5} \text{ if } v \in [0,1], \\ 3 - 2v + \frac{v^2}{5} \text{ if } v \in \left[1, \frac{5}{3}\right], \\ 2(2-v)^2 \text{ if } v \in \left[\frac{5}{3}, 2\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{4v}{5} \text{ if } v \in [0,1], \\ 1 - \frac{v}{5} \text{ if } v \in \left[1, \frac{5}{3}\right], \\ 2(2-v) \text{ if } v \in \left[\frac{5}{3}, 2\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{9}{10}$ if $P \in A \backslash B_1$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + 3B_1 + 2C_1' + D_1 + 2C_1 + D_1' + B_2$. We have $S_S(A) = \frac{10}{9}$. Thus, $\delta_P(S) \leq \frac{9}{10}$ for $P \in A$. Note that for $P \in A \backslash B_1$ we have:

$$h(v) \leq \begin{cases} \frac{8v^2}{25} \text{ if } v \in [0,1], \\ \frac{(5-v)(9v-5)}{50} \text{ if } v \in \left[1, \frac{5}{3}\right], \\ 2(2-v) \text{ if } v \in \left[\frac{5}{3}, 2\right] \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \leq \frac{5}{9} < \frac{10}{9}$. Thus, $\delta_P(S) = \frac{9}{10}$ if $P \in A \backslash B_1$. □

Lemma 8.1.26. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a). $$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(C_1 + 2B_1 + 2B_2 + C_2) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ -K_S - vA - (v-1)(C_1 + 2B_1 + 2B_2 + C_2) - (2v-3)(C_1' + C_2') \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

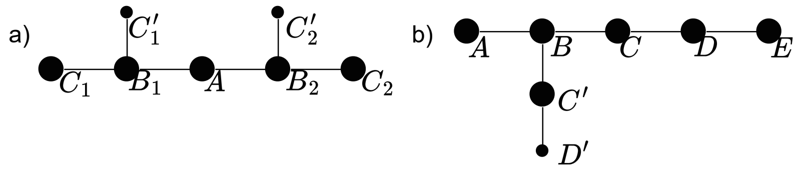


Figure 8.24: Dual graph: $(-K_S)^2=2$ and $\delta_P(S)=\frac{6}{7}$ with $-K_S-vA$ nef on $\left[\frac{3}{2},2\right]$

$$N(v)=\begin{cases}\frac{v}{3}(C_1+2B_1+2B_2+C_2) \text{ if } v\in\left[0,\frac{3}{2}\right],\\ (v-1)(C_1+2B_1+2B_2+C_2)+(2v-3)(C_1'+C_2') \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

b). $$P(v)=\begin{cases}-K_S-vA-\frac{v}{3}(2C'+4B+3C+2D+E) \text{ if } v\in\left[0,\frac{3}{2}\right],\\ -K_S-vA-(v-1)(4B+3C+2D+E)-(4v-5)C_1'-(4v-6)D' \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{3}(2C'+4B+3C+2D+E) \text{ if } v\in\left[0,\frac{3}{2}\right],\\ (v-1)(4B+3C+2D+E)+(4v-5)C_1'+(4v-6)D' \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

$$(P(v))^2=\begin{cases}2-\frac{2v^2}{3} \text{ if } v\in\left[0,\frac{3}{2}\right],\\ 2(2-v)^2 \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}\qquad P(v)\cdot A=\begin{cases}\frac{2v}{3} \text{ if } v\in\left[0,\frac{3}{2}\right],\\ 2(2-v) \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

In this case $\delta_P(S)=\frac{6}{7}$ if $P\in A$.

Proof. In part a). the Zariski Decomposition follows from $-K_S-vA\sim_{\mathbb{R}}(2-v)A+2B_1+C_1+C_1'+2B_2+C_2+C_2'$. A similar statement holds in other parts. We have $S_S(A)=\frac{7}{6}$. Thus, $\delta_P(S)\le\frac{6}{7}$ for $P\in A$. Note that for $P\in A$ we have:

$$h(v)\le\begin{cases}\frac{2v^2}{3} \text{ if } v\in\left[0,\frac{3}{2}\right],\\ 2(2-v) \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\le\frac{7}{6}$. Thus, $\delta_P(S)=\frac{6}{7}$ if $P\in A$. □

Lemma 8.1.27. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

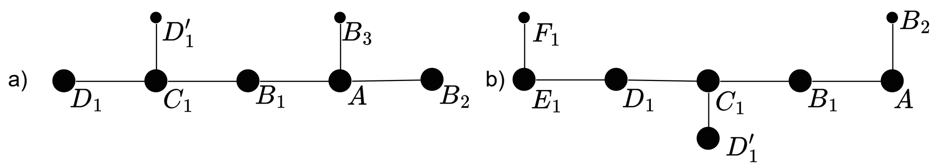


Figure 8.25: Dual graph: $(-K_S)^2=2$ and $\delta_P(S)=\frac{6}{7}$ with $-K_S-vA$ nef on $[0,1]$

Then $\tau(A)=2$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

a).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{4}(2B_2+3B_1+2C_1+D_1) \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{4}(2B_2+3B_1+2C_1+D_1)-(v-1)B_3 \text{ if } v\in[1,2].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{4}(2B_2+3B_1+2C_1+D_1) \text{ if } v\in[0,1],\\ \frac{v}{4}(2B_2+3B_1+2C_1+D_1)+(v-1)B_3 \text{ if } v\in[1,2].\end{cases}$$

b).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{4}(5B_1+6C_1+4D_1+2E_1+3D_1') \text{ if } v\in[0,1],\\ -K_S-vA-\frac{v}{4}(5B_1+6C_1+4D_1+2E_1+3D_1')-(v-1)B_2 \text{ if } v\in[1,2].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{4}(5B_1+6C_1+4D_1+2E_1+3D_1') \text{ if } v\in[0,1],\\ \frac{v}{4}(5B_1+6C_1+4D_1+2E_1+3D_1')+(v-1)B_2 \text{ if } v\in[1,2].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}2-\frac{3v^2}{4} \text{ if } v\in[0,1],\\ \frac{(v-2)(v-6)}{4} \text{ if } v\in[1,2].\end{cases}\quad P(v)\cdot A=\begin{cases}\frac{3v}{4} \text{ if } v\in[0,1],\\ 1-\frac{v}{4} \text{ if } v\in[1,2].\end{cases}$$

In this case $\delta_P(S)=\frac{6}{7}$ if $P\in A\backslash B_1$.

Proof. In part a). the Zariski Decomposition follows from $-K_S-vA\sim_{\mathbb{R}}(2-v)A+2B_1+2C_1+D_1+D_1'+B_2+B_3$. A similar statement holds in other parts. We have $S_S(A)=\frac{7}{6}$. Thus, $\delta_P(S)\le\frac{6}{7}$ for $P\in A$. Note that for $P\in A\backslash(B_1\cup B_2)$ or if $P\in A\cap B_2$ we have:

$$h(v)\le\begin{cases}\frac{9v^2}{32} \text{ if } v\in[0,1],\\ \frac{(4-v)(7v-4)}{32} \text{ if } v\in[1,2].\end{cases}\quad \text{or } h(v)\le\begin{cases}\frac{21v^2}{32} \text{ if } v\in[0,1],\\ \frac{(4-v)(3v+4)}{32} \text{ if } v\in[1,2].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\le\frac{7}{12}<\frac{7}{6}$. or $S(W^A_{\bullet,\bullet};P)\le\frac{7}{8}<\frac{7}{6}$. Thus, $\delta_P(S)=\frac{6}{7}$ if $P\in A\backslash B_1$. □

Lemma 8.1.28. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

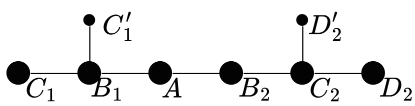


Figure 8.26: Dual graph: $(-K_S)^2=2$ and $\delta_P(S)=\frac{4}{5}$ with $\tau(A)=2$

Then $\tau(A)=2$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

$$P(v)=\begin{cases}-K_S-vA-\frac{v}{4}(3B_2+2C_2+D_2)-\frac{v}{3}(2B_1+C_1) \text{ if } v\in\left[0,\frac{3}{2}\right],\\ -K_S-vA-\frac{v}{4}(3B_2+2C_2+D_2)-(v-1)(2B_1+C_1)-(2v-3)C_1' \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(3B_2+2C_2+D_2)+\frac{v}{3}(2B_1+C_1) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{v}{4}(3B_2+2C_2+D_2)+(v-1)(2B_1+C_1)+(2v-3)C_1' \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2-\frac{7v^2}{12} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{(2-v)(10-3v)}{4} \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases} \quad P(v)\cdot A = \begin{cases} \frac{7v}{12} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ 2-\frac{3v}{4} \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{4}{5}$ *if* $P \in A$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A+2B_1+C_1+C_1'+2B_2+2C_2+D_2+D_2'$. We have $S_S(A) = \frac{5}{4}$. Thus, $\delta_P(S) \leq \frac{4}{5}$ for $P \in A$. Note that for $P \in A \backslash B_2$ or $P \in A \cap B_2$ we have:

$$h(v) \leq \begin{cases} \frac{161v^2}{288} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{(8-3v)(13v-8)}{32} \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{175v^2}{288} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{(8-3v)(3v+8)}{32} \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \leq \frac{5}{4}$. Thus, $\delta_P(S) = \frac{4}{5}$ if $P \in A$. □

Lemma 8.1.29. *Suppose* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

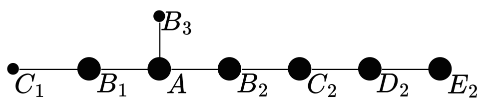


Figure 8.27: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{4}{5}$ with $\tau(A) = \frac{5}{2}$

Then $\tau(A) = \frac{5}{2}$ *and the Zariski Decomposition of the divisor* $-K_S - vA$ *is given by:*

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{5}(4B_2+3C_2+2D_2+E_2) - \frac{v}{2}B_1 \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{5}(4B_2+3C_2+2D_2+E_2) - \frac{v}{2}B_1 - (v-1)B_3 \text{ if } v \in [1,2], \\ -K_S - vA - \frac{v}{5}(4B_2+3C_2+2D_2+E_2) - (v-1)(B_1+B_3) - (v-2)C_1 \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(4B_2+3C_2+2D_2+E_2) + \frac{v}{2}B_1 \text{ if } v \in [0,2], \\ \frac{v}{5}(4B_2+3C_2+2D_2+E_2) + \frac{v}{2}B_1 + (v-1)B_3 \text{ if } v \in [1,2], \\ \frac{v}{5}(4B_2+3C_2+2D_2+E_2) + (v-1)(B_1+B_3) + (v-2)C_1 \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2-\frac{7v^2}{10} \text{ if } v\in[0,1], \\ 3-2v+\frac{3v^2}{10} \text{ if } v\in[1,2], \\ \frac{(5-2v)^2}{5} \text{ if } v\in\left[2,\frac{5}{2}\right]. \end{cases} \quad P(v)\cdot A = \begin{cases} \frac{7v}{10} \text{ if } v\in[0,1], \\ 1-\frac{3v}{10} \text{ if } v\in[1,2], \\ 2(1-\frac{2v}{5}) \text{ if } v\in\left[2,\frac{5}{2}\right]. \end{cases}$$

In this case $\delta_P(S)=\frac{4}{5}$ *if* $P\in A\backslash B_2$.

Proof. The Zariski Decomposition follows from

$$-K_S - vA \sim_{\mathbb{R}} \Big(\frac{5}{2}-v\Big)A+\frac{1}{2}\Big(4B_2+3C_2+2D_2+E_2+3B_3+3B_1+C_1\Big).$$

We have $S_S(A)=\frac{5}{4}$. Thus, $\delta_P(S)\le\frac{4}{5}$ for $P\in A$. Note that for $P\in A\backslash B_2$ we have:

$$h(v) \le \begin{cases} \frac{119v^2}{200} \text{ if } v\in[0,1], \\ \frac{(10-3v)(7v+10)}{200} \text{ if } v\in[1,2], \\ \frac{6v(5-2v)}{25} \text{ if } v\in\left[2,\frac{5}{2}\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\le\frac{53}{60}<\frac{5}{4}$. Thus, $\delta_P(S)=\frac{4}{5}$ if $P\in A\backslash B_2$. ☐

Lemma 8.1.30. *Suppose* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

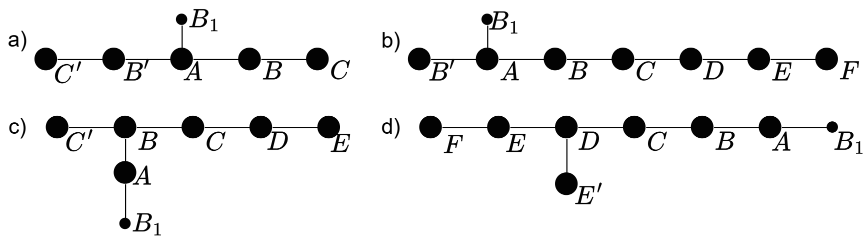


Figure 8.28: Dual graph: $(-K_S)^2=2$ and $\delta_P(S)=\frac{3}{4}$ with $\tau(A)=3$

Then $\tau(A)=3$ *and the Zariski Decomposition of the divisor* $-K_S-vA$ *is given by:*

a).
$$P(v)=\begin{cases} -K_S-vA-\frac{v}{3}(C'+2B'+2B+C) \text{ if } v\in[0,1], \\ -K_S-vA-\frac{v}{3}(C'+2B'+2B+C)-(v-1)B_1 \text{ if } v\in[1,3]. \end{cases}$$
$$N(v)=\begin{cases} \frac{v}{3}(C'+2B'+2B+C) \text{ if } v\in[0,1], \\ \frac{v}{3}(C'+2B'+2B+C)+(v-1)B_1 \text{ if } v\in[1,3]. \end{cases}$$

b).
$$P(v)=\begin{cases} -K_S-vA-\frac{v}{6}(3B'+5B+4C+3D+2E+F) \text{ if } v\in[0,1], \\ -K_S-vA-\frac{v}{6}(3B'+5B+4C+3D+2E+F)-(v-1)B_1 \text{ if } v\in[1,3]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(3B' + 5B + 4C + 3D + 2E + F) \text{ if } v \in [0,1], \\ \frac{v}{6}(3B' + 5B + 4C + 3D + 2E + F) + (v-1)B_1 \text{ if } v \in [1,3]. \end{cases}$$

c).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2C' + 4B + 3C + 2D + E) \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{3}(2C' + 4B + 3C + 2D + E) - (v-1)B_1 \text{ if } v \in [1,3]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2C' + 4B + 3C + 2D + E) \text{ if } v \in [0,1], \\ \frac{v}{3}(2C' + 4B + 3C + 2D + E) + (v-1)B_1 \text{ if } v \in [1,3]. \end{cases}$$

d).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(2F + 4E + 6D + 5C + 4B + 3E') \text{ if } v \in [0,1], \\ -K_S - vA - \frac{v}{3}(2F + 4E + 6D + 5C + 4B + 3E') - (v-1)B_1 \text{ if } v \in [1,3]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2F + 4E + 6D + 5C + 4B + 3E') \text{ if } v \in [0,1], \\ \frac{v}{3}(2F + 4E + 6D + 5C + 4B + 3E') + (v-1)B_1 \text{ if } v \in [1,3]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2 - \frac{2v^2}{3} \text{ if } v \in [0,1], \\ \frac{(3-v)^2}{3} \text{ if } v \in [1,3]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{2v}{3} \text{ if } v \in [0,1], \\ 1 - \frac{v}{3} \text{ if } v \in [1,3]. \end{cases}$$

In this case $\delta_P(S) = \frac{3}{4}$ if $P \in A$.

Proof. In part a). the Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (3-v)A + C + 2B + 2B' + C' + 2B_1$. A similar statement holds in other parts. We have $S_S(A) = \frac{4}{3}$. Thus, $\delta_P(S) \le \frac{3}{4}$ for $P \in A$. Note that if $P \in A \backslash (B \cup B')$ or if $P \in A \cap (B \cup B')$ we have:

$$h(v) \le \begin{cases} \frac{2v^2}{9} \text{ if } v \in [0,1], \\ \frac{(3-v)(5v-3)}{18} \text{ if } v \in [1,3]. \end{cases} \quad \text{or } h(v) \le \begin{cases} \frac{2v^2}{3} \text{ if } v \in [0,1], \\ \frac{(3-v)(v+1)}{6} \text{ if } v \in [1,3]. \end{cases}$$

So we have $S(W^A_{\bullet,\bullet}; P) \le \frac{2}{3} < \frac{4}{3}$ or $S(W^A_{\bullet,\bullet}; P) \le \frac{10}{9} < \frac{4}{3}$. Thus, $\delta_P(S) = \frac{3}{4}$ if $P \in A$. □

Lemma 8.1.31. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

Then $\tau(A) = 2$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).
$$P(v) = -K_S - vA - \frac{v}{4}(D_1 + 2C_1 + 3B_1 + 3B_2 + 2C_2 + D_2) \text{ if } v \in [0,2].$$
$$N(v) = \frac{v}{4}(D_1 + 2C_1 + 3B_1 + 3B_2 + 2C_2 + D_2) \text{ if } v \in [0,2].$$

b).
$$P(v) = -K_S - vA - \frac{v}{2}(B_1 + B_2 + B_3) \text{ if } v \in [0,2].$$
$$N(v) = \frac{v}{2}(B_1 + B_2 + B_3) \text{ if } v \in [0,2].$$

c).
$$P(v) = -K_S - vA - \frac{v}{2}(B_1 + 2B + C + C') \text{ if } v \in [0,2].$$

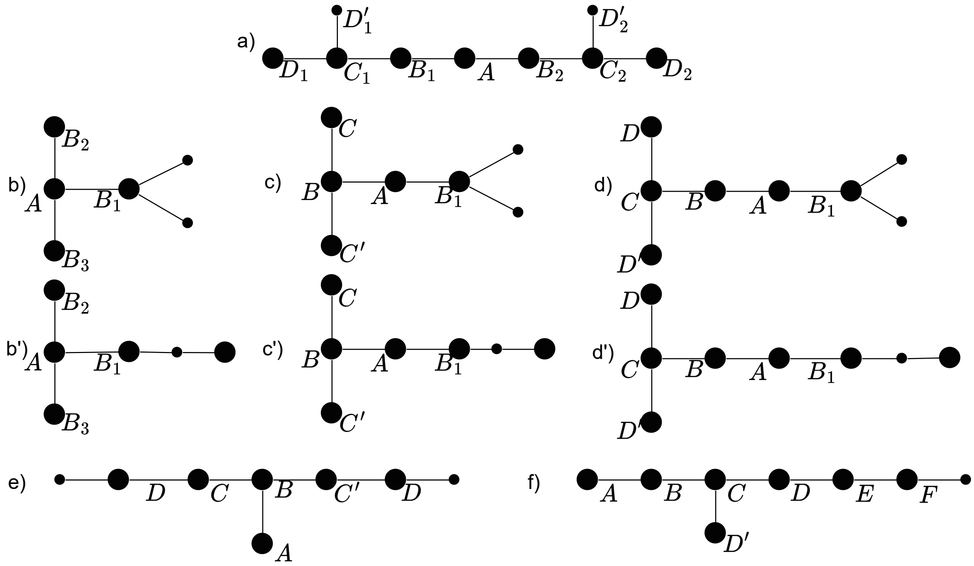


Figure 8.29: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{3}{4}$ with $\tau(A) = 2$

$$N(v) = \frac{v}{2}(B_1 + 2B + C + C') \text{ if } v \in [0,2].$$

d).
$$P(v) = -K_S - vA - \frac{v}{2}(B_1 + 2B + 2C + D + D') \text{ if } v \in [0,2].$$
$$N(v) = \frac{v}{2}(B_1 + 2B + 2C + D + D') \text{ if } v \in [0,2].$$

e).
$$P(v) = -K_S - vA - \frac{v}{2}(D + 2C + 3B + 2C' + D') \text{ if } v \in [0,2].$$
$$N(v) = \frac{v}{2}(D + 2C + 3B + 2C' + D') \text{ if } v \in [0,2].$$

f).
$$P(v) = -K_S - vA - \frac{v}{2}(3B + 4C + 3D + 2E + F + 2D') \text{ if } v \in [0,2].$$
$$N(v) = \frac{v}{2}(3B + 4C + 3D + 2E + F + 2D') \text{ if } v \in [0,2].$$

Moreover,
$$(P(v))^2 = \frac{(2-v)(2+v)}{2} P(v) \cdot A = \frac{v}{2} \text{ if } v \in [0,2].$$

In this case $\delta_P(S) = \frac{3}{4}$ if $P \in A \backslash B$.

Proof. In part a). the Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (2-v)A + D_1' + D_1 + 2C_1 + 2B_1 + 2B_2 + 2C_2 + D_2 + D_2'$. A similar statement holds in other parts. We have $S_S(A) = \frac{4}{3}$. Thus, $\delta_P(S) \leq \frac{3}{4}$ for $P \in A$. Note that for $P \in A \backslash B$ we have $h(v) = \frac{v^2}{2}$ if $v \in [0,2]$. So $S(W^A_{\bullet,\bullet}; P) = \frac{4}{3}$. Thus, $\delta_P(S) = \frac{3}{4}$ if $P \in A \backslash B$. $\square$

Lemma 8.1.32. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

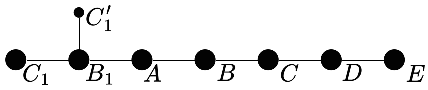


Figure 8.30: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{3}{4}$ with $\tau(A) = \frac{5}{2}$

Then $\tau(A) = \frac{5}{2}$ *and the Zariski Decomposition of the divisor* $-K_S - vA$ *is given by:*

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{5}(4B+3C+2D+E) - \frac{v}{3}(2B_1+C_1) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ -K_S - vA - \frac{v}{5}(4B+3C+2D+E) - (v-1)(2B_1+C_1) - (2v-3)C_1' \text{ if } v \in \left[\frac{3}{2}, \frac{5}{2}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(4B+3C+2D+E) + \frac{v}{3}(C_1+2B_1) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{v}{5}(4B+3C+2D+E) + (v-1)(2B_1+C_1) + (2v-3)C_1' \text{ if } v \in \left[\frac{3}{2}, \frac{5}{2}\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2 - \frac{8v^2}{15} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{(5-2v)^2}{5} \text{ if } v \in \left[\frac{3}{2}, \frac{5}{2}\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{8v}{15} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ 2(1 - \frac{2v}{5}) \text{ if } v \in \left[\frac{3}{2}, \frac{5}{2}\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{3}{4}$ *if* $P \in A\backslash B$.

Proof. The Zariski Decomposition follows from

$$-K_S - vA \sim_{\mathbb{R}} \left(\frac{5}{2} - v\right)A + \frac{1}{2}\left(4B + 3C + 2D + E + 6B_1 + 3C_1 + 4C_1'\right).$$

We have $S_S(A) = \frac{4}{3}$. Thus, $\delta_P(S) \leq \frac{3}{4}$ for $P \in A$. Note that for $P \in A\backslash B$ we have:

$$h(v) \leq \begin{cases} \frac{112v^2}{225} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{2(5-2v)(8v-5)}{25} \text{ if } v \in \left[\frac{3}{2}, \frac{5}{2}\right]. \end{cases}$$

So we have $S(W^A_{\bullet,\bullet}; P) \leq \frac{4}{3}$. Thus, $\delta_P(S) = \frac{3}{4}$ if $A\backslash B$. □

Lemma 8.1.33. *Suppose* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

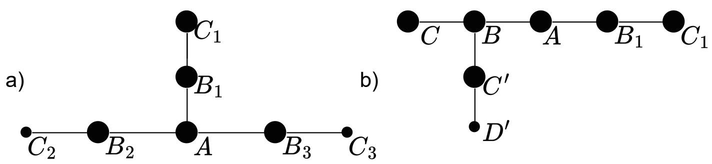


Figure 8.31: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{3}{5}$

Then $\tau(A)=3$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

a).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{6}(2C_1+4B_1+3B_2+3B_2) \text{ if } v\in[0,2],\\ -K_S-vA-\frac{v}{3}(C_1+2B_1)-(v-1)(B_2+B_3)-(v-2)(C_2+C_3) \text{ if } v\in[2,3].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{6}(2C_1+4B_1+3B_2+3B_2) \text{ if } v\in[0,2],\\ \frac{v}{3}(C_1+2B_1)+(v-1)(B_2+B_3)+(v-2)(C_2+C_3) \text{ if } v\in[2,3].\end{cases}$$

b).
$$P(v)=\begin{cases}-K_S-vA-\frac{v}{6}(3C+3C'+6B+4B_1+2C_1) \text{ if } v\in[0,2],\\ -K_S-vA-\frac{v}{3}(2B_1+C_1)-(v-1)(C+2B)-(2v-3)C'-(2v-4)D' \text{ if } v\in[2,3].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{6}(3C+3C'+6B+4B_1+2C_1) \text{ if } v\in[0,2],\\ \frac{v}{3}(2B_1+C_1)+(v-1)(C+2B)+(2v-3)C'+(2v-4)D' \text{ if } v\in[2,3].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}2-\frac{v^2}{3} \text{ if } v\in[0,2],\\ \frac{2(3-v)^2}{3} \text{ if } v\in[2,3].\end{cases}\qquad P(v)\cdot A=\begin{cases}\frac{v}{3} \text{ if } v\in[0,2],\\ 2(1-\frac{v}{3}) \text{ if } v\in[2,3].\end{cases}$$

In this case $\delta_P(S)=\frac{3}{5}$ if $P\in A\backslash B$.

Proof. In part a). the Zariski Decomposition follows from $-K_S-vA\sim_{\mathbb{R}}(3-v)A+2B_1+C_1+2B_2+C_2+2B_3+C_3$. A similar statement holds in other parts. We have $S_S(A)=\frac{5}{3}$. Thus, $\delta_P(S)\leq\frac{3}{5}$ for $P\in A$. Note that for $P\in A\cap B_1$ or if $P\in A\backslash B$ we have:

$$h(v)\leq\begin{cases}\frac{5v^2}{18} \text{ if } v\in[0,2],\\ \frac{2(3-v)(v+3)}{9} \text{ if } v\in[2,3].\end{cases}\quad\text{or } h(v)\leq\begin{cases}\frac{2v^2}{9} \text{ if } v\in[0,2],\\ \frac{4v(3-v)}{9} \text{ if } v\in[2,3].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\leq\frac{4}{3}<\frac{5}{3}$ or $S(W^A_{\bullet,\bullet};P)\leq\frac{10}{9}<\frac{5}{3}$. Thus, $\delta_P(S)=\frac{3}{5}$ if $P\in A\backslash B$. □

Lemma 8.1.34. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

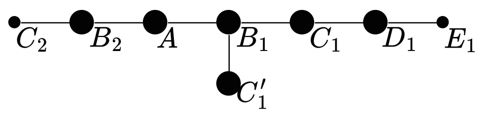


Figure 8.32: Dual graph: $(-K_S)^2=2$ and $\delta_P(S)=\frac{4}{7}$

Then $\tau(A)=3$ and the Zariski Decomposition of the divisor $-K_S-vA$ is given by:

$$P(v)=\begin{cases}-K_S-vA-\frac{v}{5}(6B_1+4C_1+2D_1+3C_1')-\frac{v}{2}B_2 \text{ if } v\in[0,2],\\ -K_S-vA-\frac{v}{5}(6B_1+4C_1+2D_1+3C_1')-(v-1)B_2-(v-2)C_2 \text{ if } v\in\left[2,\frac{5}{2}\right],\\ -K_S-vA-(v-1)(B_2+2B_1+C_1')-(v-2)C_2-(2v-3)C_1-(2v-4)D_1-(2v-5)E_1 \text{ if } v\in\left[\frac{5}{2},3\right].\end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(6B_1+4C_1+2D_1+3C_1') + \frac{v}{2}B_2 \text{ if } v \in [0,1], \\ \frac{v}{5}(6B_1+4C_1+2D_1+3C_1') + (v-1)B_2 + (v-2)C_2 \text{ if } v \in \left[2, \frac{5}{2}\right], \\ (v-1)(B_2+2B_1+C_1') + (v-2)C_2 + (2v-3)C_1 + (2v-4)D_1 + (2v-5)E_1 \text{ if } v \in \left[\frac{5}{2}, 3\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2 - \frac{3v^2}{10} \text{ if } v \in [0,2], \\ 4 - 2v + \frac{v^2}{5} \text{ if } v \in \left[2, \frac{5}{2}\right], \\ (3-v)^2 \text{ if } v \in \left[\frac{5}{2}, 3\right]. \end{cases} \quad P(v)\cdot A = \begin{cases} \frac{3v}{10} \text{ if } v \in [0,2], \\ 1 - \frac{v}{5} \text{ if } v \in \left[2, \frac{5}{2}\right], \\ 3 - v \text{ if } v \in \left[\frac{5}{2}, 3\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{4}{7}$ if $P \in A \backslash B_1$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (3-v)A + 4B_1 + 3C_1 + 2D_1 + E_1 + 2C_1' + 2B_2 + C_2$. We have $S_S(A) = \frac{7}{4}$. Thus, $\delta_P(S) \leq \frac{4}{7}$ for $P \in A$. Note that for $P \in A \backslash B_1$ we have:

$$h(v) \leq \begin{cases} \frac{39v^2}{200} \text{ if } v \in [0,2], \\ \frac{(5-v)(9v-5)}{50} \text{ if } v \in \left[2, \frac{5}{2}\right], \\ \frac{(3-v)(1+v)}{2} \text{ if } v \in \left[\frac{5}{2}, 3\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \leq \frac{7}{6} < \frac{7}{4}$. Thus, $\delta_P(S) = \frac{4}{7}$ if $P \in A \backslash B_1$. □

Lemma 8.1.35. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

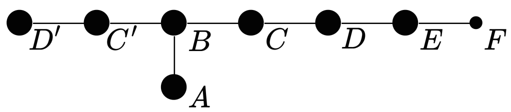


Figure 8.33: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{9}{16}$

Then $\tau(A) = 3$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{7}(4D' + 8C' + 12B + 9C + 6D + 3E) \text{ if } v \in \left[0, \frac{7}{3}\right], \\ -K_S - vA - (v-1)(D' + 2C' + 3B) - (3v-4)C - (3v-5)D - (3v-6)E - (3v-7)F \text{ if } v \in \left[\frac{7}{3}, 3\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{7}(4D' + 8C' + 12B + 9C + 6D + 3E) \text{ if } v \in \left[0, \frac{7}{3}\right], \\ (v-1)(D' + 2C' + 3B) + (3v-4)C + (3v-5)D + (3v-6)E + (3v-7)F \text{ if } v \in \left[\frac{7}{3}, 3\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2 - \frac{2v^2}{7} \text{ if } v \in \left[0, \frac{7}{3}\right], \\ (3-v)^2 \text{ if } v \in \left[\frac{7}{3}, 3\right]. \end{cases} \quad P(v)\cdot A = \begin{cases} \frac{2v}{7} \text{ if } v \in \left[0, \frac{7}{3}\right], \\ 3 - v \text{ if } v \in \left[\frac{7}{3}, 3\right]. \end{cases}$$

In this case $\delta_P(S) = \frac{9}{16}$ *if* $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (3-v)A + 2D' + 4C' + 6B + 5C + 4D + 3E + 2F$. We have $S_S(A) = \frac{16}{9}$. Thus, $\delta_P(S) \leq \frac{9}{16}$ for $P \in A$. Note that for $P \in A \backslash B$ we have:

$$h(v) \leq \begin{cases} \frac{2v^2}{49} \text{ if } v \in \left[0, \frac{7}{3}\right], \\ \frac{(3-v)^2}{2} \text{ if } v \in \left[\frac{7}{3}, 3\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \leq \frac{2}{9} < \frac{16}{9}$. Thus, $\delta_P(S) = \frac{9}{16}$ if $P \in A \backslash B$. □

Lemma 8.1.36. *Suppose* P *belongs to a* (-2)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

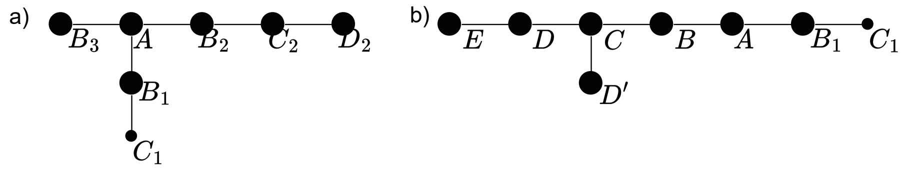


Figure 8.34: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{1}{2}$

Then $\tau(A) = 4$ *and the Zariski Decomposition of the divisor* $-K_S - vA$ *is given by:*

a).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{4}(3B_2 + 2C_2 + D_2 + 2B_3 + 2B_1) \text{ if } v \in [0,2], \\ -K_S - vA - \frac{v}{4}(3B_2 + 2C_2 + D_2 + 2B_3) - (v-1)B_1 - (v-2)C_1 \text{ if } v \in [2,4]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(3B_2 + 2C_2 + D_2 + 2B_3 + 2B_1) \text{ if } v \in [0,2], \\ \frac{v}{4}(3B_2 + 2C_2 + D_2 + 2B_3) + (v-1)B_1 + (v-2)C_1 \text{ if } v \in [2,4]. \end{cases}$$

b).

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{4}(2E + 4D + 6C + 5B + 2B_1 + 3D') \text{ if } v \in [0,2], \\ -K_S - vA - \frac{v}{4}(2E + 4D + 6C + 5B + 3D') - (v-1)B_1 - (v-2)C_1 \text{ if } v \in [2,4]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(2E + 4D + 6C + 5B + 2B_1 + 3D') \text{ if } v \in [0,2], \\ \frac{v}{4}(2E + 4D + 6C + 5B + 3D') + (v-1)B_1 + (v-2)C_1 \text{ if } v \in [2,4]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2 - \frac{v^2}{4} \text{ if } v \in [0,2], \\ \frac{(4-v)^2}{4} \text{ if } v \in [2,4]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{v}{4} \text{ if } v \in [0,2], \\ 1 - \frac{v}{4} \text{ if } v \in [2,4]. \end{cases}$$

In this case $\delta_P(S) = \frac{1}{2}$ *if* $P \in A \backslash B$.

Proof. In part a). the Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (4-v)A + 3B_2 + 2C_2 + D_2 + 2B_3 + 3B_1 + 2C_1$. A similar statement holds in other parts. We have $S_S(A) = 2$. Thus, $\delta_P(S) \le \frac{1}{2}$ for $P \in E_3$. Note that if $P \in A \cap (B \cup B')$ or if $P \in A \backslash (B \cup B')$ we have:

$$h(v) \le \begin{cases} \frac{7v^2}{32} \text{ if } v \in [0,2], \\ \frac{(4-v)(5v+4)}{32} \text{ if } v \in [2,4]. \end{cases} \quad \text{or } h(v) \le \begin{cases} \frac{5v^2}{32} \text{ if } v \in [0,2], \\ \frac{(4-v)(7v-4)}{32} \text{ if } v \in [2,4]. \end{cases}$$

So we have $S(W^A_{\bullet,\bullet};P) \le \frac{5}{3} < 2$ or $S(W^A_{\bullet,\bullet};P) \le \frac{4}{3} < 2$. Thus, $\delta_P(S) = \frac{1}{2}$ if $P \in A$. □

Lemma 8.1.37. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

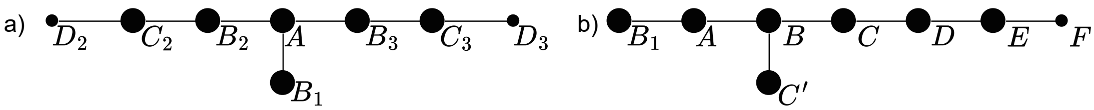


Figure 8.35: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{3}{7}$

Then $\tau(A) = 4$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

a).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{6}(3B_1 + 2C_2 + 3B_2 + 3B_3 + 2C_3) \text{ if } v \in [0,3], \\ -K_S - vA - \frac{v}{2}B_1 - (v-1)(B_2+B_3) - (v-2)(C_2+C_3) - (v-3)(D_2+D_3) \text{ if } v \in [3,4]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{6}(3B_1 + 2C_2 + 3B_2 + 3B_3 + 2C_3) \text{ if } v \in [0,3], \\ \frac{v}{2}B_1 + (v-1)(B_2+B_3) + (v-2)(C_2+C_3) + (v-3)(D_2+D_3) \text{ if } v \in [3,4]. \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vA - \frac{v}{6}(3B_1 + 8B + 6C + 4D + 2E + 4C') \text{ if } v \in [0,3], \\ -K_S - vA - \frac{v}{2}B_1 - (v-1)(2B+C') - (2v-3)C - (2v-4)D - (2v-5)E - (2v-6)F \text{ if } v \in [3,4]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{6}(3B_1 + 8B + 6C + 4D + 2E + 4C') \text{ if } v \in [0,3], \\ \frac{v}{2}B_1 + (v-1)(2B+C') + (2v-3)C + (2v-4)D + (2v-5)E + (2v-6)F \text{ if } v \in [3,4]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2 - \frac{v^2}{6} \text{ if } v \in [0,3], \\ \frac{(4-v)^2}{2} \text{ if } v \in [3,4]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{v}{6} \text{ if } v \in [0,3], \\ 2 - \frac{v}{2} \text{ if } v \in [3,4]. \end{cases}$$

In this case $\delta_P(S) = \frac{3}{7}$ if $P \in A \backslash B$.

Proof. In part a). the Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (4-v)A + 2B_1 + D_2 + 2C_2 + 3B_2 + 3B_3 + 2C_3 + D_3$. A similar statement holds in other parts. We have $S_S(A) = \frac{7}{3}$. Thus, $\delta_P(S) \le \frac{3}{7}$ for $P \in A$. Note that if $P \in A \cap B_1$ or if $P \in A\backslash(B_1 \cup B)$ we have:

$$h(v) \le \begin{cases} \frac{7v^2}{72} \text{ if } v \in [0,3], \\ \frac{(4-v)(v+4)}{8} \text{ if } v \in [3,4]. \end{cases} \quad \text{or } h(v) \le \begin{cases} \frac{v^2}{8} \text{ if } v \in [0,3], \\ \frac{3(4-v)}{8} \text{ if } v \in [3,4]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \le \frac{4}{3} < \frac{7}{3}$ or $S(W^A_{\bullet,\bullet};P) \le \frac{4}{3} < \frac{7}{3}$. Thus, $\delta_P(S) = \frac{3}{7}$ if $P \in A\backslash B$. □

Lemma 8.1.38. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

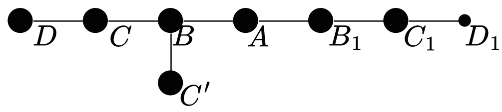


Figure 8.36: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{3}{8}$

Then $\tau(A) = 5$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{15}(6D + 12C + 18B + 9C' + 10B_1 + 5C_1) \text{ if } v \in [0,3], \\ -K_S - vA - \frac{v}{5}(2D + 4C + 6B + 3C') - (v-1)B_1 - (v-2)C_1 - (v-3)D_1 \text{ if } v \in [3,5]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{15}(6D + 12C + 18B + 9C' + 10B_1 + 5C_1) \text{ if } v \in [0,3], \\ \frac{v}{5}(2D + 4C + 6B + 3C') + (v-1)B_1 + (v-2)C_1 + (v-3)D_1 \text{ if } v \in [3,5]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2 - \frac{2v^2}{15} \text{ if } v \in [0,3], \\ \frac{(5-v)^2}{5} \text{ if } v \in [3,5]. \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{2v}{15} \text{ if } v \in [0,3], \\ 1 - \frac{v}{5} \text{ if } v \in [3,5]. \end{cases}$$

In this case $\delta_P(S) = \frac{3}{8}$ if $P \in A\backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (5-v)A + 2D + 4C + 6B + 3C' + 4B_1 + 3C_1 + 2D_1$. We have $S_S(A) = \frac{8}{3}$. Thus, $\delta_P(S) \le \frac{3}{8}$ for $P \in A$. Note that for $P \in A\backslash B$ we have:

$$h(v) \le \begin{cases} \frac{49v^2}{450} \text{ if } v \in [0,3], \\ \frac{(5-v)(9v-5)}{50} \text{ if } v \in [3,5]. \end{cases}$$

So $S(W^A_{\bullet,\bullet};P) \le \frac{21}{10} < \frac{8}{3}$. Thus, $\delta_P(S) = \frac{3}{8}$ if $P \in A\backslash B$. □

Lemma 8.1.39. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

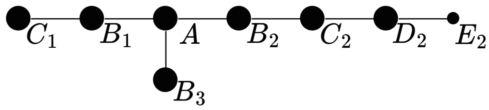


Figure 8.37: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) = \frac{3}{10}$

Then $\tau(A) = 6$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{3}(C_1 + 2B_1) - \frac{v}{2}B_3 - \frac{v}{4}(3B_2 + 2C_2 + D_2) \text{ if } v \in [0,4], \\ -K_S - vA - \frac{v}{3}(C_1 + 2B_1) - \frac{v}{2}B_3 - (v-1)B_2 - (v-2)C_2 - (v-3)D_2 - (v-4)E_2 \text{ if } v \in [4,6]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(C_1 + 2B_1) + \frac{v}{2}B_3 + \frac{v}{4}(3B_2 + 2C_2 + D_2) \text{ if } v \in [0,4], \\ \frac{v}{3}(C_1 + 2B_1) + \frac{v}{2}B_3 + (v-1)B_2 + (v-2)C_2 + (v-3)D_2 + (v-4)E_2 \text{ if } v \in [4,6]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2 - \frac{v^2}{12} \text{ if } v \in [0,4], \\ \frac{(6-v)^2}{6} \text{ if } v \in [4,6], \end{cases} \quad P(v) \cdot A = \begin{cases} \frac{v}{12} \text{ if } v \in [0,4], \\ 1 - \frac{v}{6} \text{ if } v \in [4,6]. \end{cases}$$

In this case $\delta_P(S) = \frac{3}{10}$ if $P \in A$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (6-v)A + 2C_1 + 4B_1 + 5B_2 + 4C_2 + 3D_2 + 2E_2 + 3B_3$ We have $S_S(A) = \frac{10}{3}$. Thus, $\delta_P(S) \leq \frac{3}{10}$ for $P \in A$. Note that if $P \in A \cap (B_1 \cup B_3)$ or if $P \in A \backslash (B_1 \cup B_3)$ we have:

$$h(v) \leq \begin{cases} \frac{17v^2}{288} \text{ if } v \in [0,4], \\ \frac{(6-v)(7v+6)}{72} \text{ if } v \in [4,6]. \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{19v^2}{288} \text{ if } v \in [0,4], \\ \frac{(6-v)(11v-6)}{72} \text{ if } v \in [4,6]. \end{cases}$$

So we have $S(W^A_{\bullet,\bullet};P) \leq \frac{7}{3} < \frac{10}{3}$ or $S(W^A_{\bullet,\bullet};P) \leq \frac{8}{3} < \frac{10}{3}$. Thus, $\delta_P(S) = \frac{3}{10}$ if $P \in A$. □

Lemma 8.1.40. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

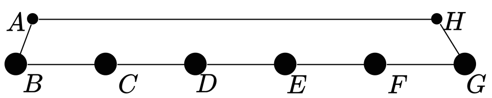


Figure 8.38: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) \geq \frac{384}{209}$

Then $\tau(A) = 1$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{7}(6B + 5C + 4D + 3E + 2F + G) \text{ if } v \in \left[0, \frac{7}{8}\right], \\ -K_S - vA - (2v-1)B - (3v-2)C - (4v-3)D - (5v-4)E - (6v-5)F - (7v-6)G - (8v-7)H \text{ if } v \in \left[\frac{7}{8}, 1\right]. \end{cases}$$

$$N(v)=\begin{cases}\frac{v}{7}(6B+5C+4D+3E+2F+G) \text{ if } v\in\left[0,\frac{7}{8}\right],\\(2v-1)B+(3v-2)C+(4v-3)D+(5v-4)E+(6v-5)F+(7v-6)G+(8v-7)H \text{ if } v\in\left[\frac{7}{8},1\right].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}2-2v-\frac{v^2}{7} \text{ if } v\in\left[0,\frac{7}{8}\right],\\9(1-v)^2 \text{ if } v\in\left[\frac{7}{8},1\right].\end{cases}\quad P(v)\cdot A=\begin{cases}1+\frac{v}{7} \text{ if } v\in\left[0,\frac{7}{8}\right],\\9(1-v) \text{ if } v\in\left[\frac{7}{8},1\right].\end{cases}$$

In this case $\delta_P(S)\geq\frac{384}{209}$ *if* $P\in A\backslash B$.

Proof. The Zariski Decomposition follows from $-K_S-vA\sim_{\mathbb{R}}(1-v)A+B+C+D+E+F+G+H$. We have $S_S(A)=\frac{23}{48}$. Thus, $\delta_P(S)\leq\frac{48}{23}$ for $P\in A$. Note that for $P\in A\backslash B$ we have:

$$h(v)\leq\begin{cases}\frac{(v+7)^2}{98} \text{ if } v\in\left[0,\frac{7}{8}\right],\\\frac{9(7v-5)(1-v)}{9} \text{ if } v\in\left[\frac{7}{8},1\right].\end{cases}$$

So $S(W^A_{\bullet,\bullet};P)\leq\frac{209}{384}$. Thus, $\delta_P(S)\geq\frac{384}{209}$ if $P\in A\backslash B$. □

Lemma 8.1.41. *Suppose* P *belongs to a* (-1)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

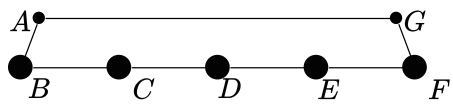


Figure 8.39: Dual graph: $(-K_S)^2=2$ and $\delta_P(S)\geq\frac{49}{27}$

Then $\tau(A)=1$ *and the Zariski Decomposition of the divisor* $-K_S-vA$ *is given by:*

$$P(v)=\begin{cases}-K_S-vA-\frac{v}{6}(5B+4C+3D+2E+F) \text{ if } v\in\left[0,\frac{6}{7}\right],\\-K_S-vA-(2v-1)B-(3v-2)C-(4v-3)D-(5v-4)E-(6v-5)F-(7v-6)G \text{ if } v\in\left[\frac{6}{7},1\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{6}(5B+4C+3D+2E+F) \text{ if } v\in\left[0,\frac{6}{7}\right],\\(2v-1)B+(3v-2)C+(4v-3)D+(5v-4)E+(6v-5)F+(7v-6)G \text{ if } v\in\left[\frac{6}{7},1\right].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}2-2v-\frac{v^2}{6} \text{ if } v\in\left[0,\frac{6}{7}\right],\\8(1-v)^2 \text{ if } v\in\left[\frac{6}{7},1\right].\end{cases}\quad P(v)\cdot A=\begin{cases}1+\frac{v}{6} \text{ if } v\in\left[0,\frac{6}{7}\right],\\8(1-v) \text{ if } v\in\left[\frac{6}{7},1\right].\end{cases}$$

In this case $\delta_P(S)\geq\frac{49}{27}$ *if* $P\in A\backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (1-v)A+B+C+D+E+F+G$. We have $S_S(A) = \frac{10}{21}$. Thus, $\delta_P(S) \le \frac{21}{10}$ for $P \in A$. Note that for $P \in A \backslash B$ we have:

$$h(v) \le \begin{cases} \frac{(v+6)^2}{72} \text{ if } v \in \left[0, \frac{6}{7}\right], \\ 8(3v-2)(1-v) \text{ if } v \in \left[\frac{6}{7}, 1\right]. \end{cases}$$

So we have $S(W^A_{\bullet,\bullet}; P) \le \frac{27}{49}$. Thus, $\delta_P(S) \ge \frac{49}{27}$ if $P \in A \backslash B$. □

Lemma 8.1.42. *Suppose P belongs to a (-1)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

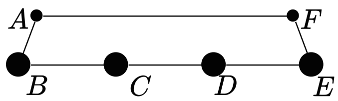


Figure 8.40: Dual graph: $(-K_S)^2 = 2$ and $\delta_P(S) \ge \frac{216}{121}$

Then $\tau(A) = 1$ and the Zariski Decomposition of the divisor $-K_S - vA$ is given by:

$$P(v) = \begin{cases} -K_S - vA - \frac{v}{5}(4B+3C+2D+E) \textit{ if } v \in \left[0, \frac{5}{6}\right], \\ -K_S - vA - (2v-1)B - (3v-2)C - (4v-3)D - (5v-4)E - (6v-5)F \textit{ if } v \in \left[\frac{5}{6}, 1\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(4B+3C+2D+E) \textit{ if } v \in \left[0, \frac{5}{6}\right], \\ (2v-1)B + (3v-2)C + (4v-3)D + (5v-4)E + (6v-5)F \textit{ if } v \in \left[\frac{5}{6}, 1\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2 - 2v - \frac{v^2}{5}, \textit{ if } v \in \left[0, \frac{5}{6}\right], \\ 7(1-v)^2 \textit{ if } v \in \left[\frac{5}{6}, 1\right]. \end{cases} \quad P(v) \cdot A = \begin{cases} 1 + \frac{v}{5} \textit{ if } v \in \left[0, \frac{5}{6}\right], \\ 7(1-v) \textit{ if } v \in \left[\frac{5}{6}, 1\right]. \end{cases}$$

In this case $\delta_P(S) \ge \frac{216}{121}$ if $P \in A \backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (1-v)A+B+C+D+E+F$. We have $S_S(A) = \frac{17}{36}$. Thus, $\delta_P(S) \le \frac{36}{17}$ for $P \in A$. Note that for $P \in A \backslash B$ we have:

$$h(v) \le \begin{cases} \frac{(v+5)^2}{50} \text{ if } v \in \left[0, \frac{5}{6}\right], \\ \frac{7(5v-3)(1-v)}{2} \text{ if } v \in \left[\frac{5}{6}, 1\right]. \end{cases}$$

So $S(W^A_{\bullet,\bullet}; P) \le \frac{121}{216}$. Thus, $\delta_P(S) \ge \frac{216}{121}$ if $P \in A \backslash B$. □

Lemma 8.1.43. *Suppose P belongs to a (-2)-curve A and there exist (-1)-curves and (-2)-curves which form the following dual graph:*

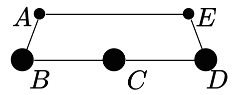


Figure 8.41: Dual graph: $(-K_S)^2=2$ and $\delta_P(S)\geq\frac{75}{43}$

Then $\tau(A)=1$ *and the Zariski Decomposition of the divisor* $-K_S-vA$ *is given by:*

$$P(v)=\begin{cases}-K_S-vA-\frac{v}{4}(3B+2C+D) \text{ if } v\in\left[0,\frac{4}{5}\right],\\ -K_S-vA-(2v-1)B-(3v-2)C-(4v-3)D-(5v-4)E \text{ if } v\in\left[\frac{4}{5},1\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{4}(3B+2C+D) \text{ if } v\in\left[0,\frac{4}{5}\right],\\ (2v-1)B+(3v-2)C+(4v-3)D+(5v-4)E \text{ if } v\in\left[\frac{4}{5},1\right].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}2-2v-\frac{v^2}{4}, \text{ if } v\in\left[0,\frac{4}{5}\right],\\ 6(1-v)^2 \text{ if } v\in\left[\frac{4}{5},1\right].\end{cases}\quad P(v)\cdot A=\begin{cases}1+\frac{v}{4} \text{ if } v\in\left[0,\frac{4}{5}\right],\\ 6(1-v) \text{ if } v\in\left[\frac{4}{5},1\right].\end{cases}$$

In this case $\delta_P(S)\geq\frac{75}{43}$ *if* $P\in A\backslash B$.

Proof. The Zariski Decomposition follows from $-K_S-vA\sim_{\mathbb{R}}(1-v)A+B+C+D+E$. We have $S_S(A)=\frac{7}{15}$. Thus, $\delta_P(S)\leq\frac{15}{7}$ for $P\in A$. Note that for $P\in A\backslash B$ we have:

$$h(v)\leq\begin{cases}\frac{(4+v)^2}{32} \text{ if } v\in\left[0,\frac{4}{5}\right],\\ 6(2v-1)(1-v) \text{ if } v\in\left[\frac{4}{5},1\right].\end{cases}$$

So we have $S(W^A_{\bullet,\bullet};P)\leq\frac{43}{75}$. Thus, $\delta_P(S)\geq\frac{75}{43}$ if $A\backslash B$. □

Lemma 8.1.44. *Suppose* P *belongs to a* (-1)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves which form the following dual graph:*

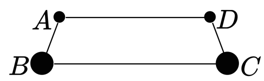


Figure 8.42: Dual graph: $(-K_S)^2=2$ and $\delta_P(S)\geq\frac{32}{19}$

Then $\tau(A)=1$ *and the Zariski Decomposition of the divisor* $-K_S-vA$ *is given by:*

$$P(v)=\begin{cases}-K_S-vA-\frac{v}{3}(2B+C) \text{ if } v\in\left[0,\frac{3}{4}\right],\\ -K_S-vA-(2v-1)B-(3v-2)C-(4v-3)D \text{ if } v\in\left[\frac{3}{4},1\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{3}(2B+C) \text{ if } v\in\left[0,\frac{3}{4}\right],\\ (2v-1)B+(3v-2)C+(4v-3)D \text{ if } v\in\left[\frac{3}{4},1\right].\end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2-2v-\frac{v^2}{3} \text{ if } v\in\left[0,\frac{3}{4}\right], \\ 5(1-v)^2 \text{ if } v\in\left[\frac{3}{4},1\right]. \end{cases} \quad P(v)\cdot A = \begin{cases} 1+\frac{v}{3} \text{ if } v\in\left[0,\frac{3}{4}\right], \\ 5(1-v) \text{ if } v\in\left[\frac{3}{4},1\right]. \end{cases}$$

In this case $\delta_P(S) \ge \frac{32}{19}$ *if* $P\in A\backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (1-v)A+B+C+D$. We have $S_S(A)=\frac{11}{24}$. Thus, $\delta_P(S)\le\frac{24}{11}$ for $P\in A$. Note that for $P\in A\backslash B$ we have:

$$h(v) \le \begin{cases} \frac{(v+3)^2}{18} \text{ if } v\in\left[0,\frac{3}{4}\right], \\ \frac{5(1-v)(3v-1)}{2} \text{ if } v\in\left[\frac{3}{4},1\right]. \end{cases}$$

So we have $S(W^A_{\bullet,\bullet};P)\le\frac{19}{32}$. Thus, $\delta_P(S)\ge\frac{32}{19}$ if $P\in A\backslash B$. □

Lemma 8.1.45. *Suppose* P *belongs to a* (-1)*-curve* A *and there exist* (-1)*-curves and* (-2)*-curves form the following dual graph:*

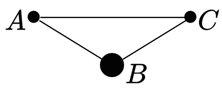


Figure 8.43: Dual graph: $(-K_S)^2=2$ and $\delta_P(S)\ge\frac{27}{17}$

Then $\tau(A)=\frac{3}{2}$ *and the Zariski Decomposition of the divisor* $-K_S-vA$ *is given by:*

$$P(v) = \begin{cases} -K_S-vA-\frac{v}{2}B \text{ if } v\in\left[0,\frac{2}{3}\right], \\ -K_S-vA-(2v-1)B-(3v-2)C \text{ if } v\in\left[\frac{2}{3},1\right], \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}B \text{ if } v\in\left[0,\frac{2}{3}\right], \\ (2v-1)B+(3v-2)C \text{ if } v\in\left[\frac{2}{3},1\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 2-2v-\frac{v^2}{2} \text{ if } v\in\left[0,\frac{2}{3}\right], \\ 4(1-v)^2 \text{ if } v\in\left[\frac{2}{3},1\right]. \end{cases} \quad P(v)\cdot A = \begin{cases} 1+\frac{v}{2} \text{ if } v\in\left[0,\frac{2}{3}\right], \\ 4(1-v) \text{ if } v\in\left[\frac{2}{3},1\right]. \end{cases}$$

In this case: $\delta_P(S) > \frac{27}{17}$ *if* $P\in A\backslash B$.

Proof. The Zariski Decomposition follows from $-K_S - vA \sim_{\mathbb{R}} (1-v)A+B+C$. We have $S_S(A) = \frac{4}{9}$. Thus, $\delta_P(S) \leq \frac{9}{4}$ for $P \in A$. Note that for $P \in A \backslash B$ we have:

$$h(v) \leq \begin{cases} \frac{(v+2)^2}{8} \text{ if } v \in \left[0, \frac{2}{3}\right], \\ 4v(1-v) \text{ if } v \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

So we have $S(W^A_{\bullet,\bullet}; P) \leq \frac{17}{27}$. Thus, $\delta_P(S) \geq \frac{27}{17}$ if $P \in A \backslash B$. □

8.2 Finding δ-invariants for degree 2

Let X be a singular del Pezzo surface of degree 2 with and S be a minimal resolution of X. Then there are several possible cases:

I. X has an $\mathbb{A}_1$ singularity and contains 44 lines,

II. X has two $\mathbb{A}_1$ singularities and contains 34 lines. In this case, we let E_1 and E_2 be the exceptional divisors, L_{12}, L'_{12} ,$L_{i,j}$ and $L'_{i,j}$ for $i \in \{1,2\}$, $j \in \{1,2,3,4\}$ be the lines on S,

III. X has three $\mathbb{A}_1$ singularities and contains 26 lines. In this case, we let E_i for $i \in \{1,2\}$ be the exceptional divisors, L_1 and L_{12} are the lines on S,

IV. X has three $\mathbb{A}_1$ singularities and contains 25 lines. In this case, we let E_1, E_2 and E_3 be the exceptional divisors, L_{123}, $L_{i,j}$ and $L'_{i,j}$ for $i \in \{1,2,3\}$, $j \in \{1,2,3,4\}$ be the lines on S,

V. X has four $\mathbb{A}_1$ singularities and contains 20 lines. In this case, we let E_1, E_2, E_3 and E_4 be the exceptional divisors, L_{ij} and L'_{ij} for $i, j \in \{1,2,3,4\}$ and $i < j$ be the lines on S,

VI. X has four $\mathbb{A}_1$ singularities and contains 19 lines. In this case, we let E_1, E_2, E_3 and E_4 be the exceptional divisors, L_{234}, $L_{1,i}$ and $L'_{1,i}$ for $i \in \{1,2,3\}$, $L_{j,k}$ and $L'_{j,k}$ for $j \in \{2,3,4\}$, $k \in \{1,2\}$ be the lines on S,

VII. X has five $\mathbb{A}_1$ singularities and contains 14 lines. In this case, we let E_1, E_2, E_3, E_4 and E_5 be the exceptional divisors, L_{134}, L_{125} L_{ij} and L'_{ij} for $(i,j) \in \{(2,3),(2,4),(3,5),(4,5)\}$, $L_{1,k}$ and $L'_{1,k}$ for $k \in \{1,2\}$ be the lines on S,

VIII. X has six $\mathbb{A}_1$ singularities and contains 10 lines. In this case, we let E_1, E_2, E_3, E_4, E_5 and E_6 be the exceptional divisors, L_{246}, L_{136}, L_{235}, L_{145}, $L_{4,5}$, L_{ij} and L'_{ij} for $(i,j) \in \{(1,2),(3,4),(5,6)\}$ be the lines on S,

IX. X has $\mathbb{A}_2$ singularity and contains 31 lines. In this case, we let E_1 and E'_1 be the exceptional divisors, $L_{1,i}$ and $L'_{1,i}$ for $i \in \{1,..,6\}$ be the lines on S,

X. X has $\mathbb{A}_2$ and $\mathbb{A}_1$ singularities and contains 20 lines. In this case, we let E_1 and E'_1, E_2 be the exceptional divisors, $L_{1,i}$ and $L'_{1,i}$ for $i \in \{1,..,4\}$, $L_{2,j}$ and $L'_{2,j}$ for $j \in \{1,2\}$, L_{12}, L'_{12} be the lines on S,

XI. X has $\mathbb{A}_2$ and two $\mathbb{A}_1$ singularities and contains 18 lines. In this case, we let E_1, E_1', E_2 and E_3 be the exceptional divisors, $L_{12}, L_{12}', L_{13}, L_{13}', L_{23}, L_{23}', L_{i,j}$ and $L_{i,j}'$ for $i,j \in \{1,2\}$ be the lines on S,

XII. X has $\mathbb{A}_2$ and three $\mathbb{A}_1$ singularities and contains 13 lines. In this case, we let E_1, E_1', E_2, E_3 and E_4 be the exceptional divisors, L_{234}, L_{12}, L_{12}', L_{13}, L_{13}', L_{14}, L_{14}', $L_{i,j}$ and $L_{i,j}'$ for $i \in \{2,3,4\}$, $j \in \{1,2\}$ be the lines on S,

XIII. X has two $\mathbb{A}_2$ singularities and contains 16 lines. In this case, we let E_1, E_1', E_2 and E_2' be the exceptional divisors, L_{12}, L_{12}'', $L_{i,j}$ and $L_{i,j}'$ for $i,j \in \{1,2\}$ be the lines on S,

XIV. X has two $\mathbb{A}_2$ and one $\mathbb{A}_1$ singularities and contains 12 lines. In this case, we let E_1, E_1', E_2, E_2' and E_3 be the exceptional divisors, L_{12}, L_{12}'', $L_{i,1}$ and $L_{i,1}'$ for $i \in \{1,2\}$ be the lines on S,

XV. X has three $\mathbb{A}_2$ singularities and contains 8 lines. In this case, we let E_1, E_1', E_2, E_2' and E_3, E_3' be the exceptional divisors, L_{ij}, for $i,j \in \{1,2,3\}$ and $i<j$ be the lines on S,

XVI. X has $\mathbb{A}_3$ singularity and contains 22 lines. In this case, we let E_1, E_1' and E_2 be the exceptional divisors, $L_{2,1}$, $L_{2,2}$, $L_{1,i}$ and $L_{1,i}'$ for $i \in \{1,2,3,4\}$ be the lines on S,

XVII. X has $\mathbb{A}_3$ and $\mathbb{A}_1$ singularities and contains 16 lines. In this case, we let E_1, E_1', E_2 and E_3 be the exceptional divisors, $L_{2,1}$, $L_{2,2}$, L_{13}, L_{13}', $L_{i,j}$ and $L_{i,j}'$ for $i \in \{1,3\}$, $j \in \{1,2\}$ be the lines on S,

XVIII. X has $\mathbb{A}_3$ and $\mathbb{A}_1$ singularities and contains 15 lines. In this case, we let E_1, E_1', E_2 and E_3 be the exceptional divisors, L_{23}, $L_{1,i}$ and $L_{1,i}'$ for $i \in \{1,2,3,4\}$, $L_{3,j}$ and $L_{3,j}'$ for $j \in \{1,2,3\}$ be the lines on S,

XIX. X has $\mathbb{A}_3$ and two $\mathbb{A}_1$ singularities and contains 12 lines. In this case, we let E_1, E_1', E_2, E_3 and E_4 be the exceptional divisors, $L_{2,1}$, $L_{2,2}$, L_{ij} and L_{ij}' for $(i,j) \in \{(1,3),(1,4),(3,4)\}$ be the lines on S,

XX. X has $\mathbb{A}_3$ and two $\mathbb{A}_1$ singularities and contains 11 lines. In this case, we let E_1, E_1', E_2, E_3 and E_4 be the exceptional divisors, L_{23}, L_{ij} and L_{ij}' for $(i,j) \in \{(1,4),(3,4)\}$, $L_{3,1}$, $L_{3,1}'$, $L_{1,k}$ and $L_{1,k}'$ for $k \in \{1,2\}$ be the lines on S,

XXI. X has $\mathbb{A}_3$ and three $\mathbb{A}_1$ singularities and contains 8 lines. In this case, we let E_1, E_1', E_2, E_3 and E_4 be the exceptional divisors, L_{345}, L_{25} L_{ij} and L_{ij}' for $(i,j) \in \{(1,4),(3,4)\}$, $L_{5,1}$, $L_{5,1}'$ be the lines on S,

XXII. X has $\mathbb{A}_3$ and $\mathbb{A}_2$ singularities and contains 10 lines. In this case, we let E_1, E_1', E_2, E_3 and E_3' be the exceptional divisors, $L_{2,1}, L_{2,2}, L_{1,1}, L_{1,1}', L_{13}, L_{13}''$ $L_{3,i}$ and $L_{3,i}'$ for $i \in \{1,2\}$ be the lines on S,

XXIII. X has $\mathbb{A}_3$, $\mathbb{A}_2$ and $\mathbb{A}_1$ singularities and contains 7 lines. In this case, we let E_1, E_1', E_2, E_3 and E_3' be the exceptional divisors, $L_{2,1}$, $L_{2,2}$, $L_{1,1}$, $L_{1,1}'$, L_{13}, L_{13}'' $L_{3,i}$ and $L_{3,i}'$ for $i \in \{1,2\}$ be the lines on S,

XXIV. X has two $\mathbb{A}_3$ singularities and contains 6 lines. In this case, we let E_1, E_1', E_2, E_3, E_3' and E_4 be the exceptional divisors, $L_{2,1}, L_{2,2}, L_{4,1}, L_{4,1}', L_{13}, L_{13}''$ be the lines on S

XXV. X has two $\mathbb{A}_3$ and on $\mathbb{A}_1$ singularities and contains 4 lines. In this case, we let E_1, E_1', E_2, E_3, E_3', E_4 and E_5 be the exceptional divisors, L_{25}, L_{45}, L_{13}, L_{13}'' be the lines on S,

XXVI. X has $\mathbb{A}_4$ singularity and contains 14 lines. In this case, we let E_1, E_1', E_2 and E_2' be the exceptional divisors, $L_{2,1}$, $L_{2,1}'$, $L_{1,i}$ and $L_{1,i}'$ for $i \in \{1,2,3\}$ be the lines on S,

XXVII. X has $\mathbb{A}_4$ and $\mathbb{A}_1$ singularities and contains 10 lines. In this case, we let E_1, E_1', E_2, E_2' and E_3 be the exceptional divisors, $L_{2,1}$, $L_{2,1}'$ $L_{1,i}$ and $L_{1,i}'$ for $i \in \{1,2,3\}$ be the lines on S,

XXVIII. X has $\mathbb{A}_4$ and $\mathbb{A}_2$ singularities and contains 6 lines. In this case, we let E_1, E_1', E_2, E_2', E_3 and E_3' be the exceptional divisors, L_{13}, L_{13}'', $L_{i,1}$ and $L_{i,1}'$ for $i \in \{2,3\}$ be the lines on S,

XXIX. X has $\mathbb{A}_5$ singularity and contains 8 lines. In this case, we let $L_{2,1}$, $L_{2,1}'$, $L_{1,i}$ and $L_{1,i}'$ for $i \in \{1,2\}$ be the lines on S,

XXX. X has $\mathbb{A}_5$ singularity and contains 7 lines. In this case, we let E_1, E_1', E_2, E_2' and E_3 be the exceptional divisors, $L_{3,1}$, $L_{1,i}$ and $L_{1,i}'$ for $i \in \{1,2,3\}$ be the lines on S,

XXXI. X has $\mathbb{A}_5$ and $\mathbb{A}_1$ singularities and contains 6 lines. In this case, we let E_1, E_1', E_2, E_2', E_3 and E_4 be the exceptional divisors, $L_{3,1}$, $L_{1,i}$ and $L_{1,i}'$ for $i \in \{1,2,3\}$ be the lines on S,

XXXII. X has $\mathbb{A}_5$ and $\mathbb{A}_1$ singularities and contains 5 lines. In this case, we let E_1, E_1', E_2, E_2', E_3 and E_4 be the exceptional divisors, $L_{3,1}$, $L_{1,i}$ and $L_{1,i}'$ for $i \in \{1,2,3\}$ be the lines on S,

XXXIII. X has $\mathbb{A}_5$ and $\mathbb{A}_2$ singularities and contains 3 lines. In this case, we let E_1, E_1', E_2, E_2', E_3, E_4 and E_4' be the exceptional divisors, L_{14}, L_{14}' and $L_{3,1}$ be the lines on S,

XXXIV. X has $\mathbb{A}_6$ singularity and contains 4 lines. In this case, we let E_1, E_1', E_2, E_2', E_3 and E_3' be the exceptional divisors, $L_{i,1}$ and $L_{i,1}'$ for $i \in \{1,2\}$ be the lines on S,

XXXV. X has $\mathbb{A}_7$ singularity and contains 2 lines. In this case, we let E_1, E_1', E_2, E_2', E_3, E_3' and E_4 be the exceptional divisors, $L_{i,1}$ and $L_{i,1}'$ for $i \in \{1,2\}$ be the lines on S,

XXXVI. X has $\mathbb{D}_4$ singularity and contains 14 lines. In this case, we let E_1, E_2, E_3, and E be the exceptional divisors, $L_{i,j}$ for $i \in \{1,2,3\}$, $j \in \{1,2\}$ be the lines on S,

XXXVII. X has $\mathbb{D}_4$ and $\mathbb{A}_1$ singularities and contains 9 lines. In this case, we let E_1, E_1, E_2, E_3 and E be the exceptional divisors, $L_{i,j}$ and $L_{i,j}'$ for $i \in \{1,3\}$, $j \in \{1,2\}$ be the lines on S,

XXXVIII. X has $\mathbb{D}_4$ and two $\mathbb{A}_1$ singularities and contains 6 lines. In this case, we let E_1, E_1', E_2, E_3 and E_3' and E be the exceptional divisors, L_i and L_i' for $i \in \{2,3\}$ be the lines on S,

XXXIX. X has $\mathbb{D}_4$ and three $\mathbb{A}_1$ singularities and contains 4 lines. In this case, we let E_1, E_1', E_2, E_2', E_3, E_3' and E be the exceptional divisors, L_i and L_i' for $i \in \{1,2,3\}$ be the lines on S,

XL. X has $\mathbb{D}_5$ singularity and contains 8 lines. In this case, we let E_1, E_1', E_2, E_3, and E be the exceptional divisors, L_1 and L_1', $L_{3,1}$ and $L_{3,2}$ be the lines on S,

XLI. X has $\mathbb{D}_5$ and $\mathbb{A}_1$ singularities and contains 5 lines. In this case, we let E_1, E_1', E_2, E_3, E_4 and E be the exceptional divisors, L_{34}, L_1 and L_1', $L_{4,1}$ and $L_{4,1}'$ be the lines on S, XLII. X has $\mathbb{D}_6$ singularity and contains 3 lines. In this case, we let E_1, E_2, E_3, E_4, E_5 and E be the exceptional divisors, L_{34}, L_5, L_5' and L be the lines on S,

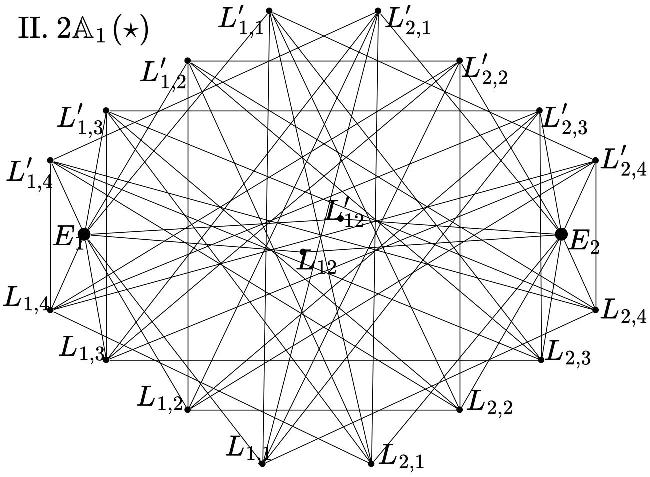
XLIII. X has $\mathbb{D}_6$ and $\mathbb{A}_1$ singularities and contains 2 lines. In this case, we let E_1, E_2, E_3, E_4, E_5, E_6 and E be the exceptional divisors, L_{56} and L be the lines on S,

XLIV. X has $\mathbb{E}_6$ singularity and contains 4 lines. In this case, we let E_1, E_1', E_2, E_2', E_3 and E be the exceptional divisors, L_1 and L_1' be the lines on S,

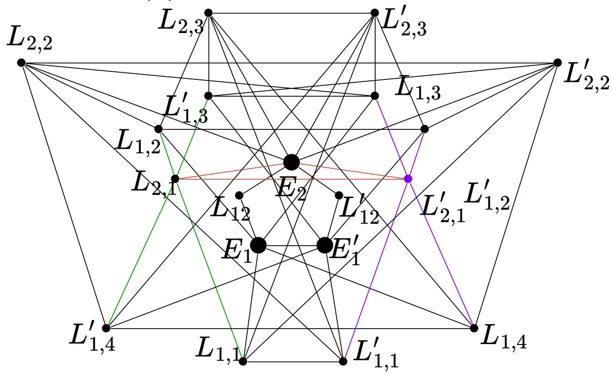
XLV. X has $\mathbb{E}_7$ singularity and contains 1 line. In this case, we let E_1, E_2, E_3, E_4, E_5, E_6 and E be the exceptional divisors, L_6 be the line on S.

such that the dual graph of the (-1)-curves and (-2)-curves or the dual graph of the (-1)-curves adjacent to a (-2)-curves if marked with $(\star)$ on S is given the picture below. Then

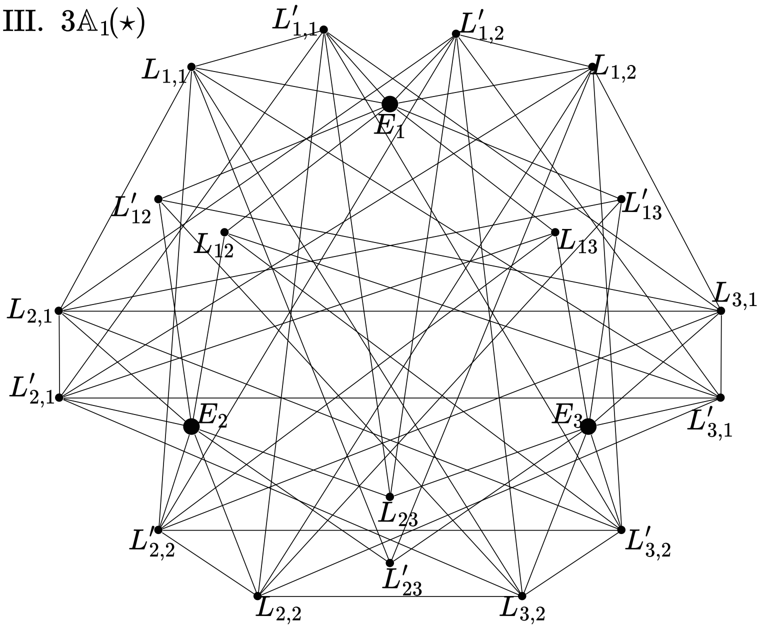
II. $2\mathbb{A}_1$ $(\star)$



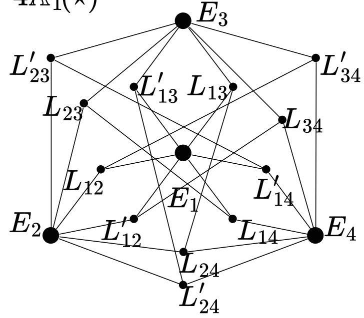
X. $\mathbb{A}_2 + \mathbb{A}_1(\star)$



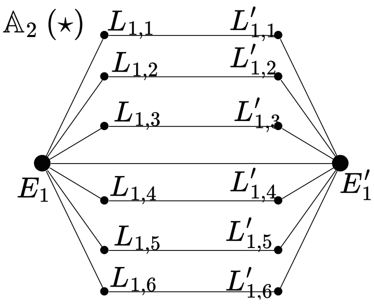
III. $3\mathbb{A}_1(\star)$



V. $4\mathbb{A}_1(\star)$



IX. $\mathbb{A}_2$ $(\star)$



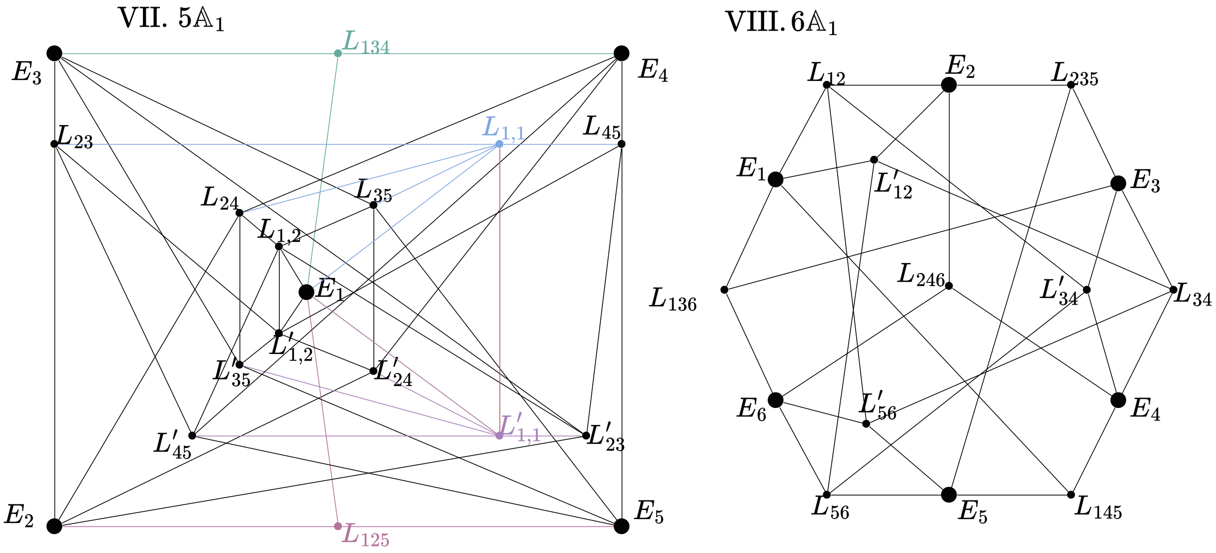
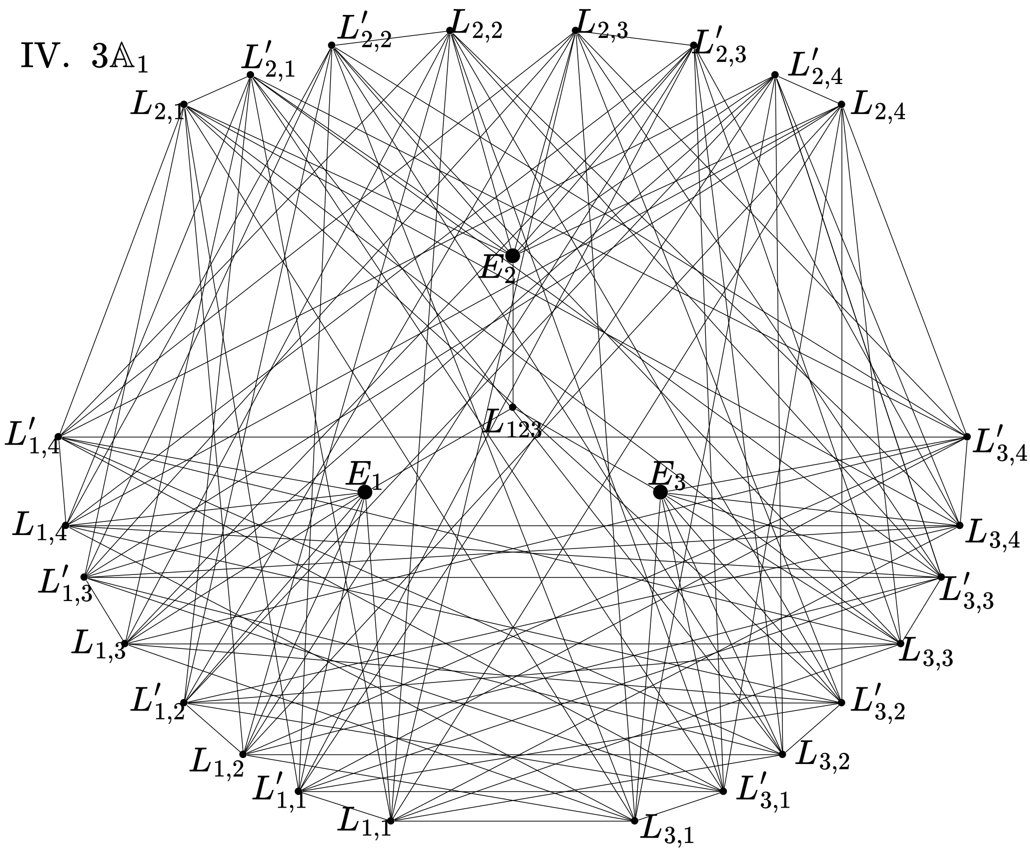


Figure 8.44: Du Val del Pezzo surfaces with $(-K_S)^2 = 2$ (pic. 1/5)



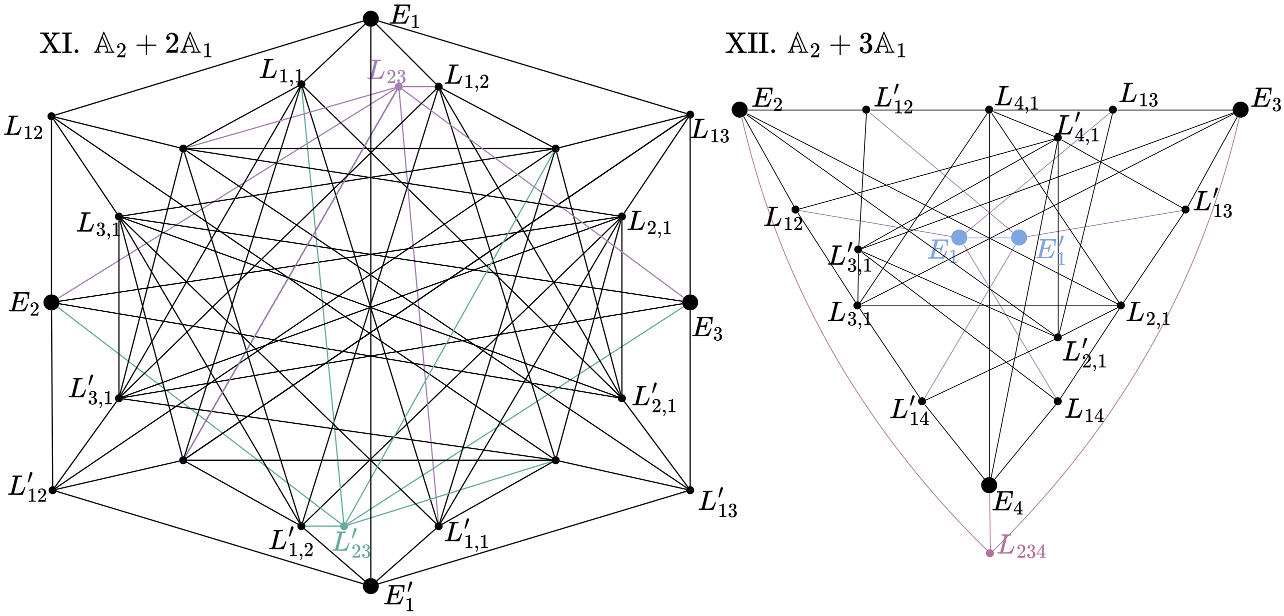
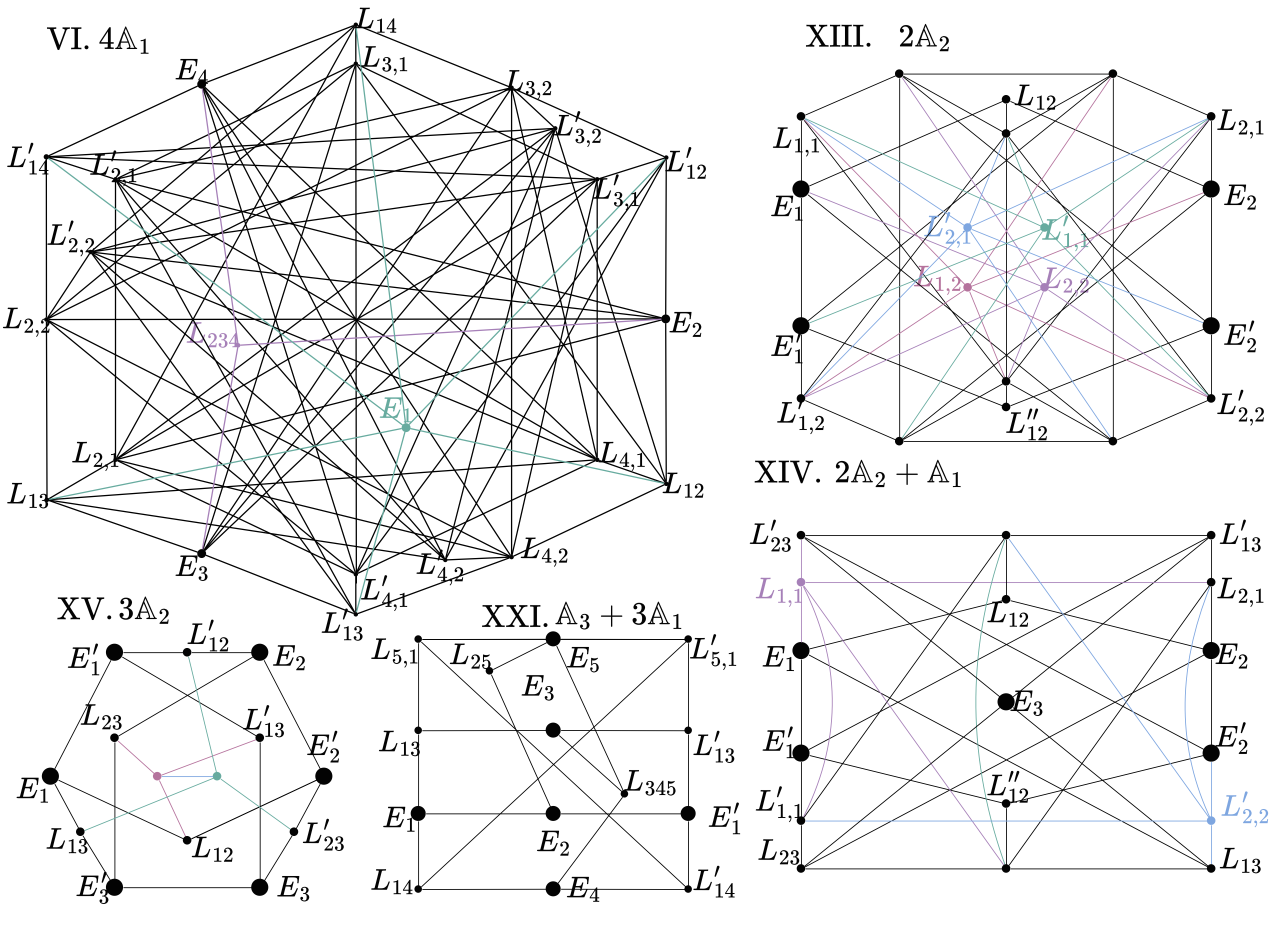


Figure 8.45: Du Val del Pezzo surfaces with $(-K_S)^2 = 2$ (pic. 2/5)



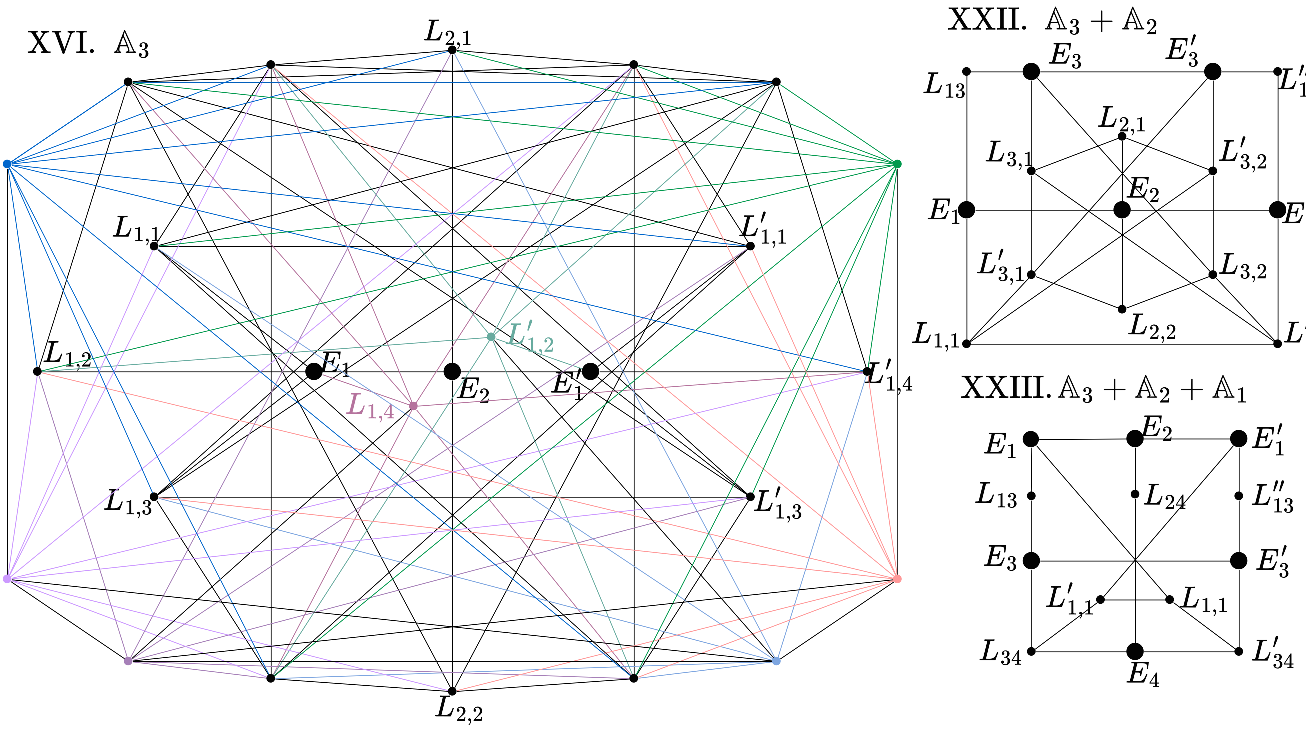
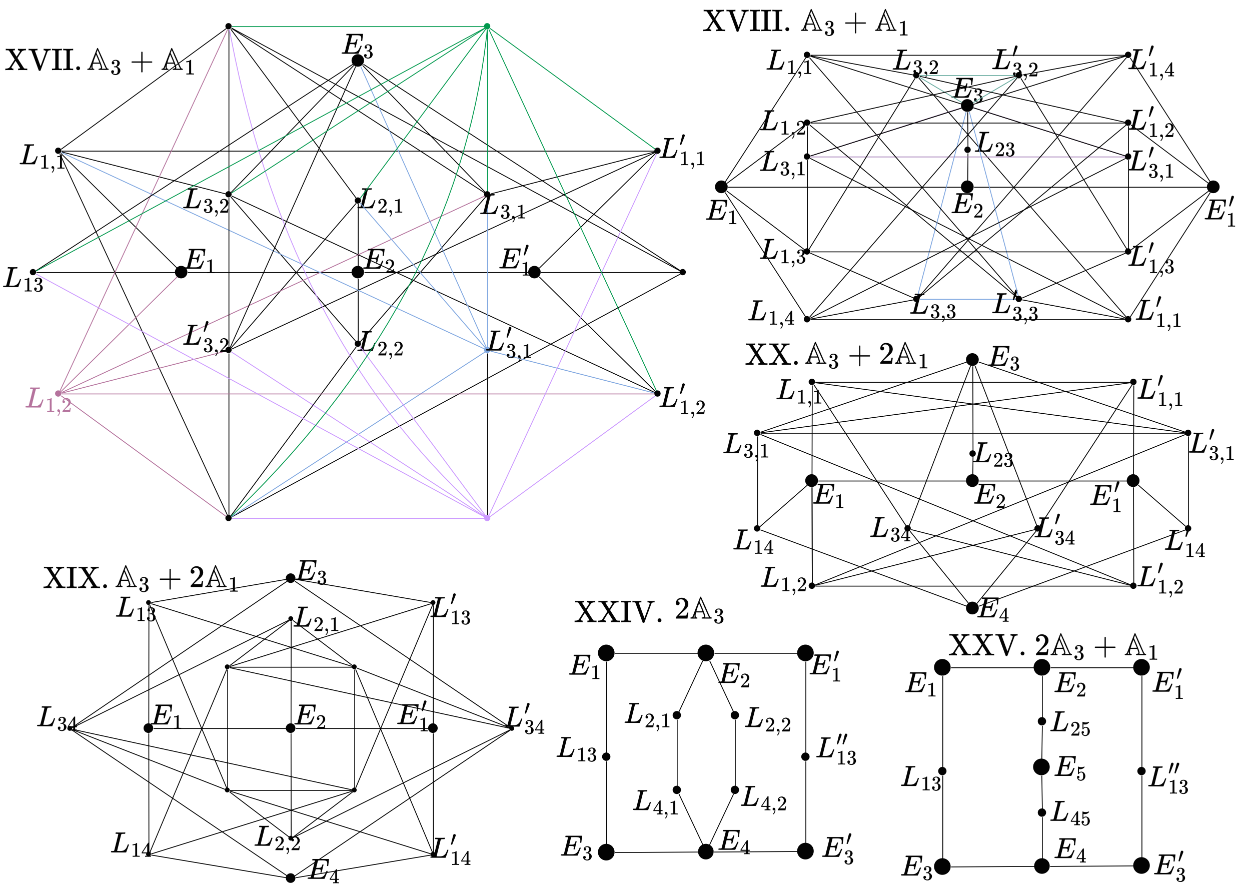


Figure 8.46: Du Val del Pezzo surfaces with $(-K_S)^2 = 2$ (pic. 3/5)



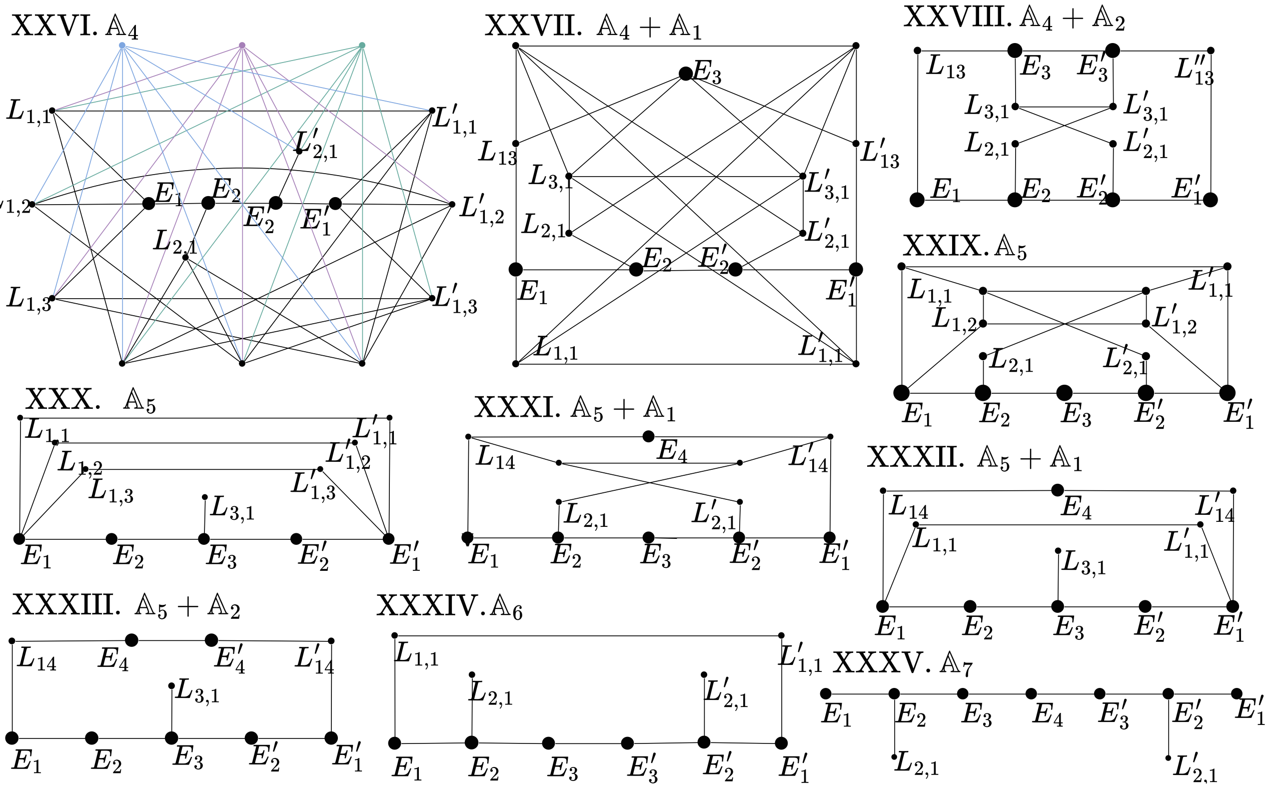


Figure 8.47: Du Val del Pezzo surfaces with $(-K_S)^2 = 2$ (pic. 4/5)

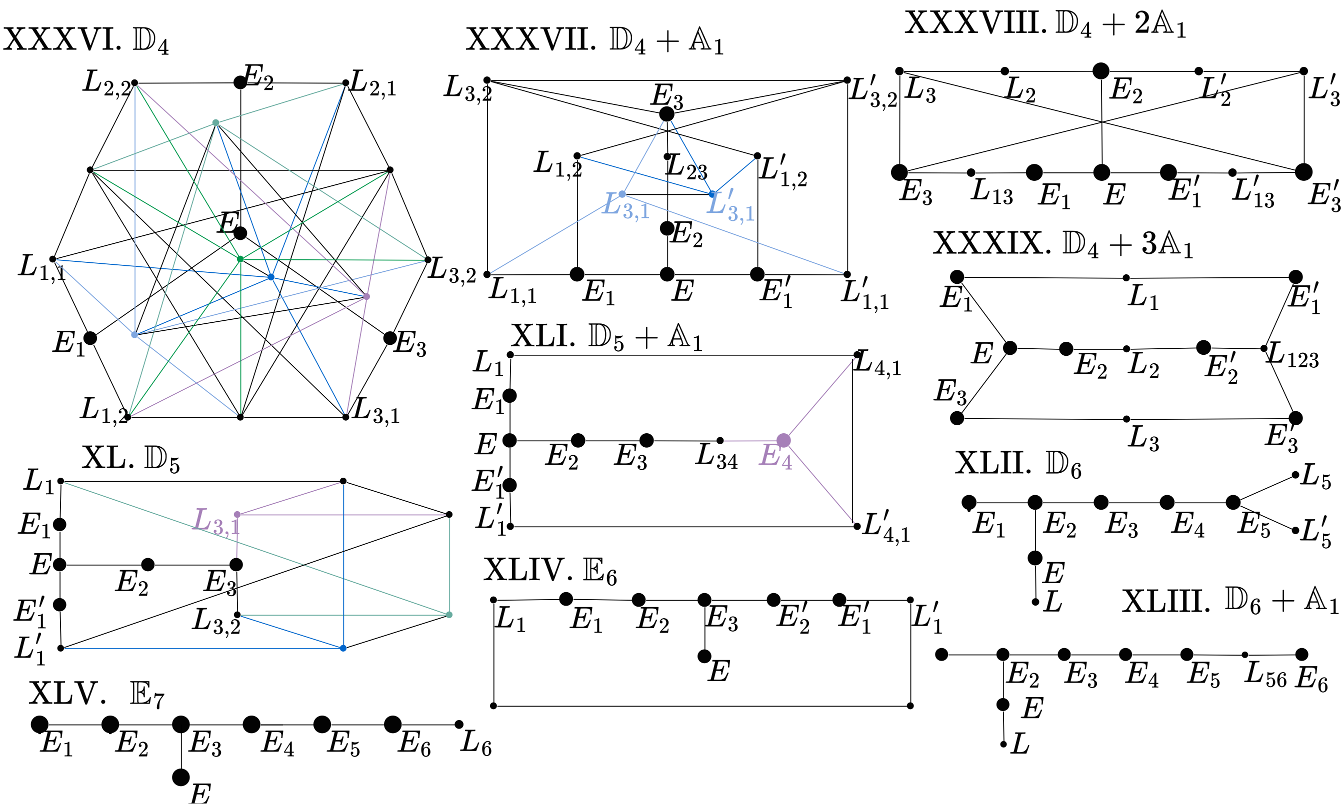


Figure 8.48: Du Val del Pezzo surfaces with $(-K_S)^2 = 2$ (pic. 5/5)

I. One has $\delta(X) = \frac{3}{2}$.

II. $\delta(X) = \frac{3}{2}$ since depending on the position of point $P \in S$ we have

P	$E_1 \cup E_2$	$(L_{12} \cup L'_{12}) \backslash (E_1 \cup E_2)$	$\mathbf{L}_2 \backslash (E_1 \cup E_2)$	o/w
$\delta_P(S)$	$\frac{3}{2}$	2	$\geq \frac{27}{17}$	$\geq \frac{9}{5}$

where $\mathbf{L}_2 := \bigcup_{i\in\{1,2\},\ j\in\{1,2,3,4\}} \left(L_{i,j} \cup L'_{i,j}\right)$.

Table 8.4: Local δ-invariants: $(-K_S)^2 = 2$ and $2\mathbb{A}_1$ singularities

III. $\delta(X) = \frac{3}{2}$ since depending on the position of point $P \in S$ we have

P	$E_1 \cup E_2 \cup E_3$	$\mathbf{L}_3^{(1)} \backslash (E_1 \cup E_2 \cup E_3)$	$\mathbf{L}_3^{(2)} \backslash (E_1 \cup E_2 \cup E_3)$	o/w
$\delta_P(S)$	$\frac{3}{2}$	2	$\geq \frac{27}{17}$	$\geq \frac{9}{5}$

where $\mathbf{L}_3^{(1)} := L_{12} \cup L_{13} \cup L_{23} \cup L'_{12} \cup L'_{13} \cup L'_{23}$, $\mathbf{L}_3^{(2)} := \bigcup_{i\in\{1,2,3\},\ j\in\{1,2\}} \left(L_{i,j} \cup L'_{i,j}\right)$.

Table 8.5: Local δ-invariants: $(-K_S)^2 = 2$ and $3\mathbb{A}_1$ singularities (26 lines)

IV. $\delta(X) = \frac{3}{2}$ since depending on the position of point $P \in S$ we have

P	$E_1 \cup E_2 \cup E_3 \cup L_{123}$	$\mathbf{L}_4 \backslash (E_1 \cup E_2 \cup E_3)$	o/w
$\delta_P(S)$	$\frac{3}{2}$	$\geq \frac{27}{17}$	$\geq \frac{9}{5}$

where $\mathbf{L}_4 := \bigcup_{i\in\{1,2,3\},\ j\in\{1,2,3,4\}} \left(L_{i,j} \cup L'_{i,j}\right) \backslash (E_1 \cup E_2 \cup E_3)$.

Table 8.6: Local δ-invariants: $(-K_S)^2 = 2$ and $3\mathbb{A}_1$ singularities (25 lines)

V. $\delta(X) = \frac{3}{2}$ since depending on the position of point $P \in S$ we have

P	$E_1 \cup E_2 \cup E_3 \cup E_4$	$\mathbf{L}_5 \backslash (E_1 \cup E_2 \cup E_3 \cup E_4)$	o/w
$\delta_P(S)$	$\frac{3}{2}$	$\geq \frac{27}{17}$	$\geq \frac{9}{5}$

where $\mathbf{L}_5 := \bigcup_{i,j\in\{1,2,3,4\},\ i<j} \left(L_{ij} \cup L'_{ij}\right) \backslash (E_1 \cup E_2 \cup E_3 \cup E_4)$.

Table 8.7: Local δ-invariants: $(-K_S)^2 = 2$ and $4\mathbb{A}_1$ singularities (20 lines)

VI. $\delta(X) = \frac{3}{2}$ since depending on the position of point $P \in S$ we have

P	$\mathbf{E}_6 \cup L_{234}$	$\bigcup_{i\in\{2,3,4\}} \left(L_{1i} \cup L'_{1i}\right) \backslash \mathbf{E}_6$	$\bigcup_{i\in\{2,3,4\},\ j\in\{1,2\}} \left(L_{i,j} \cup L'_{i,j}\right) \backslash \mathbf{E}_6$	o/w
$\delta_P(S)$	$\frac{3}{2}$	2	$\geq \frac{27}{17}$	$\geq \frac{9}{5}$

where $\mathbf{E}_6 := E_1 \cup E_2 \cup E_3 \cup E_4$.

Table 8.8: Local δ-invariants: $(-K_S)^2 = 2$ and $4\mathbb{A}_1$ singularities (19 lines)

VII. $\delta(X) = \frac{3}{2}$ since depending on the position of point $P \in S$ we have

P	$\mathbf{E}_7 \cup L_{134} \cup L_{125}$	$\mathbf{L}_7 \backslash \mathbf{E}_7$	$L_{1,1} \cup L'_{1,1} \cup L_{1,2} \cup L'_{1,2} \backslash E_1$	o/w
$\delta_P(S)$	$\frac{3}{2}$	2	$\geq \frac{27}{17}$	$\geq \frac{9}{5}$

where $\mathbf{E}_7 := E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$, $\mathbf{L}_7 := \bigcup_{(i,j)\in\{(2,3),(2,4),(3,5),(4,5)\}} \left(L_{ij} \cup L'_{ij}\right)$.

Table 8.9: Local δ-invariants: $(-K_S)^2 = 2$ and $5\mathbb{A}_1$ singularities

VIII. $\delta(X) = \frac{3}{2}$ since depending on the position of point $P \in S$ we have

P	$\mathbf{E}_8 \cup L_{246} \cup L_{136} \cup L_{235} \cup L_{145}$	$\mathbf{L}_8 \backslash \mathbf{E}_8$	o/w
$\delta_P(S)$	$\frac{3}{2}$	2	$\geq \frac{9}{5}$

where $\mathbf{E}_8 := E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6$, $\mathbf{L}_8 := \bigcup_{(i,j)\in\{(1,2),(3,4),(5,6)\}} \left(L_{ij} \cup L'_{ij}\right)$.

Table 8.10: Local δ-invariants: $(-K_S)^2 = 2$ and $6\mathbb{A}_1$ singularities

IX. $\delta(X) = \frac{6}{5}$ since depending on the position of point $P \in S$ we have

P	$E_1 \cap E'_1$	$(E_1 \cup E'_1) \backslash (E_1 \cap E'_1)$	$\mathbf{L}_9 \backslash (E_1 \cup E'_1)$	o/w
$\delta_P(S)$	$\frac{6}{5}$	$\frac{9}{7}$	$\geq \frac{32}{19}$	$\geq \frac{9}{5}$

$\mathbf{L}_9 := \bigcup_{i\in\{1,..,6\}} \left(L_{1,i} \cup L'_{1,i}\right)$.

Table 8.11: Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_2$ singularity

X. $\delta(X) = \frac{6}{5}$ since depending on the position of point $P \in S$ we have

P	$\overline{\mathbf{E}}_{10}$	$\mathbf{E}_{10} \backslash \overline{\mathbf{E}}_{10}$	E_2	$(L_{12} \cup L'_{12}) \backslash (\mathbf{E}_{10} \cup E_2)$	$\mathbf{L}^{(1)}_{10} \backslash (\mathbf{E}_{10} \cup E_2)$	$\mathbf{L}^{(2)}_{10} \backslash \mathbf{E}_{10}$	o/w
$\delta_P(S)$	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{3}{2}$	$\frac{15}{8}$	$\geq \frac{27}{17}$	$\geq \frac{32}{19}$	$\geq \frac{9}{5}$

where $\mathbf{E}_{10} := E_1 \cup E'_1$, $\overline{\mathbf{E}}_{10} := E_1 \cap E'_1$,
$\mathbf{L}^{(1)}_{10} := L_{2,1} \cup L'_{2,1} \cup L_{2,2} \cup L'_{2,2}$, $\mathbf{L}^{(2)}_{10} := \bigcup_{i\in\{1,..,4\}} \left(L_{1,i} \cup L'_{1,i}\right)$.

Table 8.12: Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_2\mathbb{A}_1$ singularities

XI. $\delta(X) = \frac{6}{5}$ since depending on the position of point $P \in S$ we have

P	$E_1 \cap E'_1$	$\mathbf{E}^{(1)}_{11} \backslash (E_1 \cap E'_1)$	$\mathbf{E}^{(2)}_{11}$	$\mathbf{L}^{(0)}_{11} \backslash (\mathbf{E}_{11} \cup \mathbf{E}^{(2)}_{11})$	$(L_{23} \cup L'_{23}) \backslash \mathbf{E}^{(2)}_{11}$
$\delta_P(S)$	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{3}{2}$	$\frac{15}{8}$	2

P	$(L_{2,1} \cup L'_{2,1} \cup L_{3,1} \cup L'_{3,1}) \backslash (\mathbf{E}^{(1)}_{11} \cup \mathbf{E}^{(2)}_{11})$	$(L_{1,1} \cup L'_{1,1} \cup L_{1,2} \cup L'_{1,2}) \backslash \mathbf{E}_{11}$	o/w
$\delta_P(S)$	$\geq \frac{27}{17}$	$\geq \frac{32}{19}$	$\geq \frac{9}{5}$

where $\mathbf{E}^{(1)}_{11} := E_1 \cup E'_1$, $\mathbf{E}^{(2)}_{11} := E_2 \cup E_3$, $\mathbf{L}^{(0)}_{11} := L_{12} \cup L'_{12} \cup L_{13} \cup L'_{13}$.

Table 8.13: Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_2 2\mathbb{A}_1$ singularities

XII. $\delta(X) = \frac{6}{5}$ since depending on the position of point $P \in S$ we have

P	$E_1 \cap E'_1$	$\mathbf{E}^{(1)}_{12} \backslash (E_1 \cap E'_1)$	$\mathbf{E}^{(2)}_{12} \cup L_{234}$	$\mathbf{L}^{(0)}_{12} \backslash (\mathbf{E}^{(1)}_{12} \cup \mathbf{E}^{(2)}_{12})$	$\mathbf{L}^{(1)}_{12} \backslash \mathbf{E}^{(2)}_{12}$	o/w
$\delta_P(S)$	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{3}{2}$	$\frac{15}{8}$	$\geq \frac{27}{17}$	$\geq \frac{9}{5}$

where $\mathbf{E}^{(1)}_{12} := E_1 \cup E'_1$, $\mathbf{E}^{(2)}_{12} := E_2 \cup E_3 \cup E_4$, $\mathbf{L}^{(0)}_{12} := L_{12} \cup L'_{12} \cup L_{13} \cup L'_{13} \cup L_{14} \cup L'_{14}$,
$\mathbf{L}^{(1)}_{12} := \bigcup_{i\in\{2,3,4\},\, j\in\{1,2\}} \left(L_{i,j} \cup L'_{i,j}\right)$.

Table 8.14: Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_2 3\mathbb{A}_1$ singularities

XIII. $\delta(X) = \frac{6}{5}$ since depending on the position of point $P \in S$ we have

P	$\mathbf{E}_{13}^{(0)}$	$\mathbf{E}_{13} \backslash \mathbf{E}_{13}^{(0)}$	$(L_{12} \cup L''_{12}) \backslash \mathbf{E}_{13}$	$\bigcup_{i,j \in \{1,2\}} (L_{i,j} \cup L'_{i,j}) \backslash \mathbf{E}_{13}$	o/w
$\delta_P(S)$	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{12}{7}$	$\geq \frac{32}{19}$	$\geq \frac{9}{5}$

where $\mathbf{E}_{13}^{(0)} := (E_1 \cap E'_1) \cup (E_2 \cap E'_2)$, $\mathbf{E}_{13} := E_1 \cup E'_1 \cup E_2 \cup E'_2$.

Table 8.15: Local δ-invariants: $(-K_S)^2 = 2$ and $2\mathbb{A}_2$ singularities

XIV. $\delta(X) = \frac{6}{5}$ since depending on the position of point $P \in S$ we have

P	$\mathbf{E}_{14}^{(0)}$	$\mathbf{E}_{14} \backslash \mathbf{E}_{14}^{(0)}$	E_3	$(L_{12} \cup L''_{12}) \backslash \mathbf{E}_{14}$	$\mathbf{L}_{14}^{(1)} \backslash (\mathbf{E}_{14} \cup E_3)$	$\mathbf{L}_{14}^{(2)} \backslash \mathbf{E}_{14}$	o/w
$\delta_P(S)$	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{3}{2}$	$\frac{12}{7}$	$\frac{15}{8}$	$\geq \frac{32}{19}$	$\geq \frac{9}{5}$

where $\mathbf{E}_{14}^{(0)} := (E_1 \cap E'_1) \cup (E_2 \cap E'_2)$, $\mathbf{E}_{14} := E_1 \cup E'_1 \cup E_2 \cup E'_2$,
$\mathbf{L}_{14}^{(1)} := L_{13} \cup L'_{13} \cup L_{23} \cup L'_{23}$, $\mathbf{L}_{14}^{(2)} := L_{1,1} \cup L'_{1,1} \cup L_{2,1} \cup L'_{2,1}$.

Table 8.16: Local δ-invariants: $(-K_S)^2 = 2$ and $2\mathbb{A}_2\mathbb{A}_1$ singularities

XV. $\delta(X) = \frac{6}{5}$ since depending on the position of point $P \in S$ we have

P	$\mathbf{E}_{15}^{(0)}$	$\mathbf{E}_{15} \backslash \mathbf{E}_{15}^{(0)}$	$\bigcup_{i,j \in \{1,2,3\},\, i<j} (L_{ij} \cup L'_{ij}) \backslash \mathbf{E}_{15}$	o/w
$\delta_P(S)$	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{12}{7}$	$\geq \frac{9}{5}$

where $\mathbf{E}_{15}^{(0)} := (E_1 \cap E'_1) \cup (E_2 \cap E'_2) \cup (E_3 \cap E'_3)$, $\mathbf{E}_{15} := E_1 \cup E'_1 \cup E_2 \cup E'_2 \cup E_3 \cup E'_3$.

Table 8.17: Local δ-invariants: $(-K_S)^2 = 2$ and $3\mathbb{A}_2$ singularities

XVI. $\delta(X) = 1$ since depending on the position of point $P \in S$ we have

P	E_2	$(E_1 \cup E'_1) \backslash E_2$	$(L_{2,1} \cup L_{2,2}) \backslash E_2$	$\bigcup_{i \in \{1,2,3,4\}} (L_{1,i} \cup L'_{1,i}) \backslash (E_1 \cup E'_1)$	o/w
$\delta_P(S)$	1	$\frac{6}{5}$	2	$\geq \frac{75}{43}$	$\geq \frac{9}{5}$

Table 8.18: Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_3$ singularity

XVII. $\delta(X) = 1$ since depending on the position of point $P \in S$ we have

P	E_2	$(E_1 \cup E'_1) \backslash E_2$	E_3	$(L_{13} \cup L'_{13}) \backslash (E_1 \cup E'_1 \cup E_3)$	$(L_{2,1} \cup L_{2,2}) \backslash E_2$
$\delta_P(S)$	1	$\frac{6}{5}$	$\frac{3}{2}$	$\frac{9}{5}$	$\frac{15}{8}$

P	$(L_{3,1} \cup L'_{3,1} \cup L_{3,2} \cup L'_{3,2}) \backslash E_3$	$(L_{1,1} \cup L'_{1,1} \cup L_{1,2} \cup L'_{1,2}) \backslash (E_1 \cup E'_1)$	o/w
$\delta_P(S)$	$\geq \frac{27}{17}$	$\geq \frac{75}{43}$	$\geq \frac{9}{5}$

Table 8.19: Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_3\mathbb{A}_1$ singularities (16 lines)

XVIII. $\delta(X) = 1$ since depending on the position of point $P \in S$ we have

P	E_2	$(E_1 \cup E_1')\backslash E_2$	$E_3 \cup L_{23}$	$\mathbf{L}_{18}^{(1)}\backslash E_3$	$\mathbf{L}_{18}^{(2)}\backslash (E_1 \cup E_1')$	o/w
$\delta_P(S)$	1	$\frac{6}{5}$	$\frac{3}{2}$	$\geq \frac{27}{17}$	$\geq \frac{75}{43}$	$\geq \frac{9}{5}$

where $\mathbf{L}_{18}^{(1)} := \bigcup_{i\in\{1,2,3\}} \left(L_{3,i} \cup L_{3,i}'\right)$, $\mathbf{L}_{18}^{(2)} := \bigcup_{j\in\{1,2,3,4\}} \left(L_{1,j} \cup L_{1,j}'\right)$.

Table 8.20: Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_3\mathbb{A}_1$ singularities (15 lines)

XIX. $\delta(X) = 1$ since depending on the position of point $P \in S$ we have

P	E_2	$\mathbf{E}_{19}\backslash E_2$	$E_3 \cup E_4$	$\mathbf{L}_{19}^{(1)}\backslash (\mathbf{E}_{19} \cup E_3 \cup E_4)$	$\mathbf{L}_{19}^{(2)}\backslash (E_2 \cup E_3 \cup E_4)$	o/w
$\delta_P(S)$	1	$\frac{6}{5}$	$\frac{3}{2}$	$\frac{9}{5}$	2	$\geq \frac{9}{5}$

where $\mathbf{E}_{19} := E_1 \cup E_1'$, $\mathbf{L}_{19}^{(1)} := L_{13} \cup L_{13}' \cup L_{14} \cup L_{14}'$, $\mathbf{L}_{19}^{(2)} := L_{34} \cup L_{34}' \cup L_{2,1} \cup L_{2,2}$.

Table 8.21: Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_3 2\mathbb{A}_1$ singularities (12 lines)

XX. $\delta(X) = 1$ since depending on the position of point $P \in S$ we have

P	E_2	$(E_1 \cup E_1')\backslash E_2$	$(E_3 \cup E_4 \cup L_{23})\backslash E_2$	$(L_{14} \cup L_{14}')\backslash (E_1 \cup E_1' \cup E_4)$
$\delta_P(S)$	1	$\frac{6}{5}$	$\frac{3}{2}$	$\frac{9}{5}$

P	$(L_{34} \cup L_{34}')\backslash (E_3 \cup E_4)$	$\mathbf{L}_{20}^{(1)}\backslash E_3$	$\mathbf{L}_{20}^{(2)}\backslash (E_1 \cup E_1')$	o/w
$\delta_P(S)$	2	$\geq \frac{27}{17}$	$\geq \frac{75}{43}$	$\geq \frac{9}{5}$

where $\mathbf{L}_{20}^{(1)} := L_{3,1} \cup L_{3,1}$, $\mathbf{L}_{20}^{(2)} := \bigcup_{k\in\{1,2\}} \left(L_{1,k} \cup L_{1,k}'\right)$.

Table 8.22: Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_3 2\mathbb{A}_1$ singularities (11 lines)

XXI. $\delta(X) = 1$ since depending on the position of point $P \in S$ we have

P	E_2	$\mathbf{E}_{21}^{(1)}\backslash E_2$	$(\mathbf{E}_{21}^{(2)} \cup E_5 \cup L_{345} \cup L_{25})\backslash E_2$	$\mathbf{L}_{21}^{(1)}\backslash (\mathbf{E}_{21}^{(1)} \cup \mathbf{E}_{21}^{(2)})$	$\mathbf{L}_{21}^{(2)}\backslash E_5$	o/w
$\delta_P(S)$	1	$\frac{6}{5}$	$\frac{3}{2}$	$\frac{9}{5}$	$\geq \frac{27}{17}$	$\geq \frac{9}{5}$

where $\mathbf{E}_{21}^{(1)} := E_1 \cup E_1'$, $\mathbf{E}_{21}^{(2)} := E_3 \cup E_4$, $\mathbf{L}_{21}^{(1)} := L_{13} \cup L_{13}' \cup L_{14} \cup L_{14}'$, $\mathbf{L}_{21}^{(2)} := L_{5,1} \cup L_{5,1}'$.

Table 8.23: Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_3 3\mathbb{A}_1$ singularities

XXII. $\delta(X) = 1$ since depending on the position of point $P \in S$ we have

P	E_2	$(\mathbf{E}_{22}^{(0)} \cup E_1 \cup E_1')\backslash E_2$	$(E_3 \cup E_3')\backslash \mathbf{E}_{22}^{(0)}$	$(L_{13} \cup L_{13}'')\backslash \mathbf{E}_{22}$	$(L_{2,1} \cup L_{2,2})\backslash E_2$
$\delta_P(S)$	1	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{18}{11}$	2

P	$(L_{3,1} \cup L_{3,1}' \cup L_{3,2} \cup L_{3,2}')\backslash \mathbf{E}_{22}$	$(L_{1,1} \cup L_{1,1}')\backslash (E_1 \cup E_1')$	o/w
$\delta_P(S)$	$\geq \frac{27}{17}$	$\geq \frac{75}{43}$	$\geq \frac{9}{5}$

where $\mathbf{E}_{22}^{(0)} := E_3 \cap E_3'$, $\mathbf{E}_{22} := E_1 \cup E_1' \cup E_3 \cup E_3'$.

Table 8.24: Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{A}_3\mathbb{A}_2$ singularities

XXIII. $\delta(X)=1$ since depending on the position of point $P\in S$ we have

P	E_2	$(\mathbf{E}_{23}^{(0)}\cup E_1\cup E_1')\setminus E_2$	$(E_3\cup E_3')\setminus\mathbf{E}_{23}^{(0)}$	$(L_{13}\cup L_{13}'')\setminus\mathbf{E}_{23}$	$(E_4\cup L_{24})\setminus E_2$
$\delta_P(S)$	1	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{18}{11}$	$\frac{3}{2}$

P	$(L_{34}\cup L_{34}')\setminus(E_3\cup E_3'\cup E_4)$	$\mathbf{L}_{23}\setminus\mathbf{E}_{23}$	$(L_{1,1}\cup L_{1,1}')\setminus(E_1\cup E_1')$	o/w
$\delta_P(S)$	$\frac{15}{8}$	$\geq\frac{27}{17}$	$\geq\frac{75}{43}$	$\geq\frac{9}{5}$

where $\mathbf{E}_{23}^{(0)}:=E_3\cap E_3'$, $\mathbf{E}_{23}^{(1)}:=E_1\cup E_1'\cup E_3\cup E_3'$, $\mathbf{L}_{23}:=L_{3,1}\cup L_{3,1}'\cup L_{3,2}\cup L_{3,2}'$.

Table 8.25: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{A}_3\mathbb{A}_2\mathbb{A}_1$ singularities

XXIV. $\delta(X)=1$ since depending on the position of point $P\in S$ we have

P	$E_2\cup E_4$	$\mathbf{E}_{24}\setminus(E_2\cup E_4)$	$(L_{13}\cup L_{13}'')\setminus\mathbf{E}_{24}$	$\mathbf{L}_{24}\setminus(E_2\cup E_4)$	o/w
$\delta_P(S)$	1	$\frac{6}{5}$	$\frac{3}{2}$	2	$\geq\frac{9}{5}$

where $\mathbf{E}_{24}:=E_1\cup E_1'\cup E_3\cup E_3'$, $\mathbf{L}_{24}:=L_{2,1}\cup L_{2,2}\cup L_{4,1}\cup L_{4,2}$.

Table 8.26: Local δ-invariants: $(-K_S)^2=2$ and $2\mathbb{A}_3$ singularities

XXV. $\delta(X)=1$ since depending on the position of point $P\in S$ we have

P	$E_2\cup E_4$	$\mathbf{E}_{25}\setminus(E_2\cup E_4)$	$(L_{13}\cup L_{13}''\cup E_5\cup L_{25}\cup L_{45})\setminus\mathbf{E}_{25}$	o/w
$\delta_P(S)$	1	$\frac{6}{5}$	$\frac{3}{2}$	$\geq\frac{9}{5}$

where $\mathbf{E}_{25}:=E_1\cup E_1'\cup E_3\cup E_3'$.

Table 8.27: Local δ-invariants: $(-K_S)^2=2$ and $2\mathbb{A}_3\mathbb{A}_1$ singularities

XXVI. $\delta(X)=\frac{12}{13}$ since depending on the position of point $P\in S$ we have

P	$E_2\cup E_2'$	$(E_1\cup E_1')\setminus(E_2\cup E_2')$	$(L_{2,1}\cup L_{2,1}')\setminus(E_2\cup E_2')$	$\bigcup_{i\in\{1,2,3\}}\left(L_{1,i}\cup L_{1,i}'\right)$	o/w
$\delta_P(S)$	$\frac{12}{13}$	$\frac{36}{31}$	$\frac{24}{13}$	$\geq\frac{216}{121}$	$\geq\frac{9}{5}$

Table 8.28: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{A}_4$ singularity

XXVII. $\delta(X)=\frac{12}{13}$ since depending on the position of point $P\in S$ we have

P	$E_2\cup E_2'$	$(E_1\cup E_1')\setminus(E_2\cup E_2')$	E_3	$(L_{2,1}\cup L_{2,1}')\setminus(E_2\cup E_2')$
$\delta_P(S)$	$\frac{12}{13}$	$\frac{36}{31}$	$\frac{3}{2}$	$\frac{24}{13}$

P	$(L_{13}\cup L_{13}')\setminus(E_1\cup E_1'\cup E_3)$	$(L_{3,1}\cup L_{3,1}')\setminus E_3$	$(L_{1,1}\cup L_{1,1}')\setminus(E_1\cup E_1')$	o/w
$\delta_P(S)$	$\frac{72}{41}$	$\geq\frac{27}{17}$	$\geq\frac{216}{21}$	$\geq\frac{9}{5}$

Table 8.29: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{A}_4\mathbb{A}_1$ singularities

XXVIII. $\delta(X)=\frac{12}{13}$ since depending on the position of point $P\in S$ we have

P	$E_2\cup E_2'$	$(E_1\cup E_1')\backslash(E_2\cup E_2')$	$E_3\cap E_3'$	$(E_3\cup E_3)\backslash(E_3\cap E_3)$
$\delta_P(S)$	$\frac{12}{13}$	$\frac{36}{31}$	$\frac{6}{5}$	$\frac{9}{7}$

P	$(L_{13}\cup L_{13}')\backslash(E_1\cup E_1'\cup E_3\cup E_3')$	$(L_{2,1}\cup L_{2,1}')\backslash(E_2\cup E_2')$	$(L_{3,1}\cup L_{3,1}')\backslash E_3$	o/w
$\delta_P(S)$	$\frac{36}{23}$	$\frac{24}{13}$	$\geq\frac{32}{19}$	$\geq\frac{9}{5}$

Table 8.30: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{A}_4\mathbb{A}_2$ singularities

XXIX. $\delta(X)=\frac{6}{7}$ since depending on the position of point $P\in S$ we have

P	$E_2\cup E_2'\cup E_3$	$\mathbf{E}_{29}^{(1)}\backslash\mathbf{E}_{29}^{(2)}$	$(L_{2,1}\cup L_{2,1}')\backslash\mathbf{E}_{29}^{(2)}$	$\bigcup_{i\in\{1,2\}}(L_{1,i}\cup L_{1,i}')\backslash\mathbf{E}_{29}^{(1)}$	o/w
$\delta_P(S)$	$\frac{6}{7}$	$\frac{8}{7}$	$\frac{12}{7}$	$\geq\frac{49}{27}$	$\geq\frac{9}{5}$

where $\mathbf{E}_{29}^{(1)}:=E_1\cup E_1'$, $\mathbf{E}_{29}^{(2)}:=E_2\cup E_2'$.

Table 8.31: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{A}_5$ singularity (8 lines)

XXX. $\delta(X)=\frac{3}{4}$ since depending on the position of point $P\in S$ we have

P	E_3	$\mathbf{E}_{30}^{(2)}\backslash E_3$	$\mathbf{E}_{30}^{(1)}\backslash\mathbf{E}_{30}^{(2)}$	$L_{3,1}\backslash E_3$	$\bigcup_{i\in\{1,2,3\}}(L_{1,i}\cup L_{1,i}')\backslash\mathbf{E}_{30}^{(1)}$	o/w
$\delta_P(S)$	$\frac{3}{4}$	$\frac{9}{10}$	$\frac{9}{8}$	$\frac{3}{2}$	$\geq\frac{49}{27}$	$\geq\frac{9}{5}$

where $\mathbf{E}_{30}^{(1)}:=E_1\cup E_1'$, $\mathbf{E}_{30}^{(2)}:=E_2\cup E_2'$.

Table 8.32: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{A}_5$ singularity (7 lines)

XXXI. $\delta(X)=\frac{6}{7}$ since depending on the position of point $P\in S$ we have

P	$E_2\cup E_2'\cup E_3$	$\mathbf{E}_{31}^{(1)}\backslash\mathbf{E}_{31}^{(2)}$	E_4	$(L_{2,1}\cup L_{2,1}'\cup L_{14}\cup L_{14}')\backslash(\mathbf{E}_{31}^{(1)}\cup\mathbf{E}_{31}^{(2)}\cup E_4)$	o/w
$\delta_P(S)$	$\frac{6}{7}$	$\frac{8}{7}$	$\frac{3}{2}$	$\frac{12}{7}$	$\geq\frac{9}{5}$

where $\mathbf{E}_{31}^{(1)}:=E_1\cup E_1'$, $\mathbf{E}_{31}^{(2)}:=E_2\cup E_2'$.

Table 8.33: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{A}_5\mathbb{A}_1$ singularities (6 lines)

XXXII. $\delta(X)=\frac{3}{4}$ since depending on the position of point $P\in S$ we have

P	E_3	$\mathbf{E}_{32}^{(2)}\backslash E_3$	$\mathbf{E}_{32}^{(1)}\backslash\mathbf{E}_{32}^{(2)}$	$(E_4\cup L_{3,1})\backslash E_3$	$\mathbf{L}_{32}^{(1)}\backslash(\mathbf{E}_{32}^{(1)}\cup E_4)$	$\mathbf{L}_{32}^{(2)}\backslash\mathbf{E}_{32}^{(1)}$	o/w
$\delta_P(S)$	$\frac{3}{4}$	$\frac{9}{10}$	$\frac{9}{8}$	$\frac{3}{2}$	$\frac{45}{26}$	$\geq\frac{49}{27}$	$\geq\frac{9}{5}$

where $\mathbf{E}_{32}^{(1)}:=E_1\cup E_1'$, $\mathbf{E}_{32}^{(2)}:=E_2\cup E_2'$, $\mathbf{L}_{32}^{(1)}:=L_{14}\cup L_{14}'$, $\mathbf{L}_{32}^{(2)}:=L_{1,1}\cup L_{1,1}$.

Table 8.34: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{A}_5\mathbb{A}_1$ singularities (5 lines)

XXXIII. $\delta(X)=\frac{3}{4}$ since depending on the position of point $P\in S$ we have

P	E_3	$\mathbf{E}_{33}^{(2)}\backslash E_3$	$\mathbf{E}_{33}^{(1)}\backslash\mathbf{E}_{33}^{(2)}$	$\mathbf{E}_{33}^{(0)}$	$\mathbf{E}_{33}^{(4)}\backslash\mathbf{E}_{33}^{(0)}$	$\mathbf{L}_{33}\backslash(\mathbf{E}_{33}^{(1)}\cup E_3\cup\mathbf{E}_{33}^{(4)})$	o/w
$\delta_P(S)$	$\frac{3}{4}$	$\frac{9}{10}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{3}{2}$	$\geq\frac{9}{5}$

where $\mathbf{E}_{33}^{(0)}:=E_4\cap E_4'$, $\mathbf{E}_{33}^{(i)}:=E_i\cup E_i'$ $(i\in\{1,2,4\})$, $\mathbf{L}_{33}:=L_{3,1}\cup L_{14}\cup L_{14}'$.

Table 8.35: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{A}_5\mathbb{A}_2$ singularities

XXXIV. $\delta(X)=\frac{4}{5}$ since depending on the position of point $P\in S$ we have

P	$\mathbf{E}_{34}^{(2)}\cup\mathbf{E}_{34}^{(3)}$	$\mathbf{E}_{34}^{(1)}\backslash\mathbf{E}_{34}^{(2)}$	$(L_{2,1}\cup L_{2,1}')\backslash\mathbf{E}_{34}^{(2)}$	$(L_{1,1}\cup L_{1,1}')\backslash\mathbf{E}_{34}^{(1)}$	o/w
$\delta_P(S)$	$\frac{4}{5}$	$\frac{60}{53}$	$\frac{60}{37}$	$\geq\frac{384}{209}$	$\geq\frac{9}{5}$

where $\mathbf{E}_{34}^{(1)}:=E_1\cup E_1'$, $\mathbf{E}_{34}^{(2)}:=E_2\cup E_2'$, $\mathbf{E}_{34}^{(3)}:=E_3\cup E_3'$.

Table 8.36: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{A}_6$ singularity

XXXV. $\delta(X)=\frac{3}{4}$ since depending on the position of point $P\in S$ we have

P	$\mathbf{E}_{35}^{(2)}\cup\mathbf{E}_{35}^{(3)}\cup E_4$	$\mathbf{E}_{35}^{(1)}\backslash\mathbf{E}_{35}^{(2)}$	$(L_{2,1}\cup L_{2,1}')\backslash\mathbf{E}_{35}^{(2)}$	o/w
$\delta_P(S)$	$\frac{3}{4}$	$\frac{9}{8}$	$\frac{3}{2}$	$\geq\frac{9}{5}$

where $\mathbf{E}_{35}^{(1)}:=E_1\cup E_1'$, $\mathbf{E}_{35}^{(2)}:=E_2\cup E_2'$, $\mathbf{E}_{35}^{(3)}:=E_3\cup E_3'$.

Table 8.37: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{A}_7$ singularity

XXXVI. $\delta(X)=\frac{3}{4}$ since depending on the position of point $P\in S$ we have

P	E	$(E_1\cup E_2\cup E_3)\backslash E$	$\bigcup_{i\in\{1,2,3\},j\in\{1,2\}}L_{i,j}\backslash(E_1\cup E_2\cup E_3)$	o/w
$\delta_P(S)$	$\frac{3}{4}$	1	2	$\geq\frac{9}{5}$

Table 8.38: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{D}_4$ singularity

XXXVII. $\delta(X)=\frac{3}{4}$ since depending on the position of point $P\in S$ we have

P	E	$(\mathbf{E}_{37}^{(1)}\cup E_2)\backslash E$	$(E_3\cup L_{23})\backslash E_2$	$\mathbf{L}_{37}^{(1)}\backslash\mathbf{E}_{37}^{(1)}$	$\mathbf{L}_{37}^{(2)}\backslash E_3$	o/w
$\delta_P(S)$	$\frac{3}{4}$	1	$\frac{3}{2}$	2	$\geq\frac{27}{17}$	$\geq\frac{9}{5}$

where $\mathbf{E}_{37}^{(1)}:=E_1\cup E_1'$, $\mathbf{L}_{37}^{(1)}:=\bigcup_{i\in\{1,2\}}\left(L_{1,i}\cup L_{1,i}'\right)$, $\mathbf{L}_{37}^{(2)}:=L_{3,1}\cup L_{3,1}'\cup L_{3,2}\cup L_{3,2}'$.

Table 8.39: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{D}_4\mathbb{A}_1$ singularities

XXXVIII. $\delta(X)=\frac{3}{4}$ since depending on the position of point $P\in S$ we have

P	E	$(\mathbf{E}_{38}^{(1)}\cup E_2)\backslash E$	$(\mathbf{E}_{38}^{(3)}\cup L_{13}\cup L_{13}')\backslash \mathbf{E}_{38}^{(1)}$	$\mathbf{L}_{38}\backslash(E_2\cup\mathbf{E}_{38}^{(3)})$	o/w
$\delta_P(S)$	$\frac{3}{4}$	1	$\frac{3}{2}$	2	$\geq\frac{9}{5}$

where $\mathbf{E}_{38}^{(1)}:=E_1\cup E_1'$, $\mathbf{E}_{38}^{(3)}:=E_3\cup E_3'$, $\mathbf{L}_{38}:=L_2\cup L_2'\cup L_3\cup L_3'$.

Table 8.40: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{D}_4 2\mathbb{A}_1$ singularities

XXXIX. $\delta(X)=\frac{3}{4}$ since depending on the position of point $P\in S$ we have

P	E	$\mathbf{E}_{39}\backslash E$	$(\mathbf{E}_{39}'\cup L_1\cup L_2\cup L_3\cup L_{123})\backslash\mathbf{E}_{39}$	o/w
$\delta_P(S)$	$\frac{3}{4}$	1	$\frac{3}{2}$	$\geq\frac{9}{5}$

where $\mathbf{E}_{39}:=E_1\cup E_2\cup E_3$, $\mathbf{E}_{39}':=E_1'\cup E_2'\cup E_3'$.

Table 8.41: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{D}_4 3\mathbb{A}_1$ singularities

XL. $\delta(X)=\frac{3}{5}$ since depending on the position of point $P\in S$ we have

P	E	$E_2\backslash E$	$\mathbf{E}_{40}\backslash E$	$E_3\backslash E_2$	$(L_1\cup L_1')\backslash\mathbf{E}_{40}$	$(L_{3,1}\cup L_{3,2})\backslash E_3$	o/w
$\delta_P(S)$	$\frac{3}{5}$	$\frac{3}{4}$	$\frac{9}{10}$	1	$\frac{9}{5}$	2	$\geq\frac{9}{5}$

where $\mathbf{E}_{40}:=E_1\cup E_1'$.

Table 8.42: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{D}_5$ singularity

XLI. $\delta(X)=\frac{3}{5}$ since depending on the position of point $P\in S$ we have

P	E	$E_2\backslash E$	$\mathbf{E}_{41}\backslash E$	$E_3\backslash E_2$	$(E_4\cup L_{34})\backslash E_3$	$\mathbf{L}_{41}\backslash\mathbf{E}_{41}$	$(L_{4,1}\cup L_{4,1}')\backslash E_4$	o/w
$\delta_P(S)$	$\frac{3}{5}$	$\frac{3}{4}$	$\frac{9}{10}$	1	$\frac{3}{2}$	$\frac{9}{5}$	$\geq\frac{27}{17}$	$\geq\frac{9}{5}$

where $\mathbf{E}_{41}:=E_1\cup E_1'$, $\mathbf{L}_{41}:=L_1\cup L_1'$.

Table 8.43: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{D}_5\mathbb{A}_1$ singularities

XLII. $\delta(X)=\frac{1}{2}$ since depending on the position of point $P\in S$ we have

P	E_2	$E_3\backslash E_2$	$(E\cup E_4)\backslash(E_2\cup E_3)$	$E_1\backslash E_2$	$E_4\backslash E_5$	$L\backslash E$	$(L_5\cup L_5')\backslash E_5$	o/w
$\delta_P(S)$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{3}{4}$	$\frac{6}{7}$	1	$\frac{3}{2}$	2	$\geq\frac{9}{5}$

Table 8.44: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{D}_6$ singularity

XLIII. $\delta(X)=\frac{1}{2}$ since depending on the position of point $P\in S$ we have

P	E_2	$E_3\backslash E_2$	$(E\cup E_4)\backslash(E_2\cup E_3)$	$E_1\backslash E_2$	$E_4\backslash E_5$	$(L\cup E_6\cup L_{56})\backslash E_5$	o/w
$\delta_P(S)$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{3}{4}$	$\frac{6}{7}$	1	$\frac{3}{2}$	$\geq\frac{9}{5}$

Table 8.45: Local δ-invariants: $(-K_S)^2=2$ and $\mathbb{D}_6\mathbb{A}_1$ singularities

XLIV. $\delta(X) = \frac{3}{7}$ since depending on the position of point $P \in S$ we have

P	E_3	$(E_2 \cup E_2')\backslash E_3$	$E\backslash E_3$	$(E_1 \cup E_1')\backslash(E_2 \cup E_2')$	$(L_1 \cup L_1')\backslash(E_1 \cup E_1')$	o/w
$\delta_P(S)$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{3}{4}$	$\frac{6}{7}$	$\frac{12}{7}$	$\geq \frac{9}{5}$

Table 8.46: Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{E}_6$ singularity

XLV. $\delta(X) = \frac{3}{10}$ since depending on the position of point $P \in S$ we have

P	E_3	$E_4\backslash E_3$	$E_2\backslash E_3$	$E_5\backslash E_4$	$E\backslash E_3$	$(E_1 \cup E_6)\backslash(E_2 \cup E_5)$	$L_6\backslash E_6$	o/w
$\delta_P(S)$	$\frac{3}{10}$	$\frac{3}{8}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{9}{16}$	$\frac{3}{4}$	$\frac{3}{2}$	$\geq \frac{9}{5}$

Table 8.47: Local δ-invariants: $(-K_S)^2 = 2$ and $\mathbb{E}_7$ singularity

Proof. We prove each case separately using lemmas from the previous section.

I. If P is a point on the unique (-2)-curve the assertion follows from the Lemma 8.1.12. If P on all the curves adjacent to the unique (-2)-curve, the assertion follows from Lemma 8.1.45. Otherwise, the assertion follows from Lemma 8.1.1.

II. If $P \in E_1 \cup E_2$, the assertion follows from Lemma 8.1.12. If $P \in (L_{12} \cup L_{12}')\backslash(E_1 \cup E_2)$, the assertion follows from Lemma 8.1.3 [a).]. If $P \in \bigcup_{i\in\{1,2\},\ j\in\{1,2,3,4\}} \left(L_{i,j} \cup L_{i,j}'\right)\backslash(E_1 \cup E_2)$, the assertion follows from Lemma 8.1.45. Otherwise, the assertion follows from Lemma 8.1.1.

III. If $P \in E_1 \cup E_2 \cup E_3$, the assertion follows from Lemma 8.1.12. If $P \in (L_{12} \cup L_{13} \cup L_{23} \cup L_{12}' \cup L_{13}' \cup L_{23}')\backslash(E_1 \cup E_2 \cup E_3)$, the assertion follows from Lemma 8.1.3 [a).]. If $P \in \bigcup_{i\in\{1,2,3\},\ j\in\{1,2\}} \left(L_{i,j} \cup L_{i,j}'\right)\backslash(E_1 \cup E_2 \cup E_3)$, the assertion follows from Lemma 8.1.45. Otherwise, the assertion follows from Lemma 8.1.1.

IV. If $P \in E_1 \cup E_2 \cup E_3$, the assertion follows from Lemma 8.1.12. If $P \in L_{123}\backslash(E_1 \cup E_2 \cup E_3)$, the assertion follows from Lemma 8.1.13 [a).]. If $P \in \bigcup_{i\in\{1,2,3\},\ j\in\{1,2,3,4\}} \left(L_{i,j} \cup L_{i,j}'\right)\backslash(E_1 \cup E_2 \cup E_3)$, the assertion follows from Lemma 8.1.45. Otherwise, the assertion follows from Lemma 8.1.1.

V. If $P \in E_1 \cup E_2 \cup E_3 \cup E_4$, the assertion follows from Lemma 8.1.12. If $P \in \bigcup_{i,j\in\{1,2,3,4\},\ i<j} \left(L_{ij} \cup L_{ij}'\right)\backslash(E_1 \cup E_2 \cup E_3 \cup E_4) \bigcup_{i,j\in\{1,2,3,4\},\ i<j} \left(L_{ij} \cup L_{ij}'\right)\backslash(E_1 \cup E_2 \cup E_3 \cup E_4)$, the assertion follows from Lemma 8.1.3 [a).]. Otherwise, the assertion follows from Lemma 8.1.1.

VI. If $P \in E_1 \cup E_2 \cup E_3 \cup E_4$, the assertion follows from Lemma 8.1.12. If $P \in L_{234}\backslash(E_1 \cup E_2 \cup E_3 \cup E_4)$, the assertion follows from Lemma 8.1.13[a).]. If $P \in \bigcup_{i\in\{2,3,4\},\ i<j} \left(L_{1i} \cup L_{1i}'\right)\backslash(E_1 \cup E_2 \cup E_3 \cup E_4)$, the assertion follows from Lemma 8.1.3 [a).]. If $P \in \bigcup_{i\in\{2,3,4\},\ j\in\{1,2\}} \left(L_{i,j} \cup L_{i,j}'\right)\backslash(E_2 \cup E_3 \cup E_4)$, the assertion follows from Lemma 8.1.45. Otherwise, the assertion follows from Lemma 8.1.1.

VII. If $P \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$, the assertion follows from Lemma 8.1.12. If $P \in (L_{134} \cup L_{125}) \backslash (E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5)$, the assertion follows from Lemma 8.1.13 [a).]. If $P \in \bigcup_{(i,j)\in\{(2,3),(2,4),(3,5),(4,5)\}} \left(L_{ij} \cup L'_{ij}\right) \backslash (E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5)$, the assertion follows from Lemma 8.1.3 [a).]. If $P \in (L_{1,1} \cup L'_{1,1} \cup L_{1,2} \cup L'_{1,2}) \backslash E_1$, the assertion follows from Lemma 8.1.45. Otherwise, the assertion follows from Lemma 8.1.1.

VIII. If $P \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6$, the assertion follows from Lemma 8.1.12. If $P \in (L_{136} \cup L_{235} \cup L_{145}) \backslash (E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6)$, the assertion follows from Lemma 8.1.13 [a).]. If $P \in \bigcup_{(i,j)\in\{(1,2),(3,4),(5,6)\}} \left(L_{ij} \cup L'_{ij}\right) \backslash (E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6)$, the assertion follows from Lemma 8.1.3 [a).]. Otherwise, the assertion follows from Lemma 8.1.1.

IX. If $P \in E_1 \cap E_1$, the assertion follows from Lemma 8.1.15. If $P \in (E_1 \cup E'_1) \backslash (E_1 \cap E'_1)$, the assertion follows from Lemma 8.1.14 [a).]. If $P \in \bigcup_{i\in\{1,..,6\}} \left(L_{1,i} \cup L'_{1,i}\right) \backslash (E_1 \cup E'_1)$, the assertion follows from Lemma 8.1.44. Otherwise, the assertion follows from Lemma 8.1.1.

X. If $P = E_1 \cap E'_1$, the assertion follows from Lemma 8.1.15. If $P \in (E_1 \cup E'_1) \backslash (E_1 \cap E'_1)$, the assertion follows from Lemma 8.1.14 [b).]. If $P \in \bigcup_{i\in\{1,..,4\}} \left(L_{1,i} \cup L'_{1,i}\right) \backslash (E_1 \cup E'_1)$, the assertion follows from Lemma 8.1.44. If $P \in E_2$, the assertion follows from Lemma 8.1.12. If $P \in (L_{12} \cup L'_{12}) \backslash (E_1 \cup E'_1 \cup E_2)$, the assertion follows from Lemma 8.1.4 [a).]. If $P \in (L_{2,1} \cup L'_{2,1} \cup L_{2,2} \cup L'_{2,2}) \backslash (E_1 \cup E'_1 \cup E_2)$, the assertion follows from Lemma 8.1.45. Otherwise, the assertion follows from Lemma 8.1.1.

XI. If $P = E_1 \cap E'_1$, the assertion follows from Lemma 8.1.15. If $P \in (E_1 \cup E'_1) \backslash (E_1 \cap E'_1)$, the assertion follows from Lemma 8.1.14 [c).]. If $P \in E_2 \cup E_3$, the assertion follows from Lemma 8.1.12. If $P \in (L_{12} \cup L'_{12} \cup L_{13} \cup L'_{13}) \backslash (E_1 \cup E'_1 \cup E_2 \cup E_3)$, the assertion follows from Lemma 8.1.4 [b).]. If $P \in (L_{23} \cup L'_{23}) \backslash (E_2 \cup E_3)$. By Lemma 8.1.3 [a).]. If $P \in (L_{2,1} \cup L'_{2,1} \cup L_{3,1} \cup L'_{2,2}) \backslash (E_2 \cup E_3)$, the assertion follows from Lemma 8.1.45. If $P \in (L_{1,1} \cup L'_{1,1} \cup L_{1,2} \cup L'_{1,2}) \backslash (E_1 \cup E'_1)$, the assertion follows from Lemma 8.1.44. Otherwise, the assertion follows from Lemma 8.1.1.

XII. If $P = E_1 \cap E'_1$, the assertion follows from Lemma 8.1.15. If $P \in (E_1 \cup E'_1) \backslash (E_1 \cap E'_1)$, the assertion follows from Lemma 8.1.14 [d).]. If $P \in E_2 \cup E_3 \cup E_4$, the assertion follows from Lemma 8.1.12. If $P \in L_{234} \backslash (E_2 \cup E_3 \cup E_4)$. By Lemma 8.1.13 [a).]. If $P \in (L_{12} \cup L'_{12} \cup L_{13} \cup L'_{13} \cup L_{14} \cup L'_{14}) \backslash (E_1 \cup E'_1 \cup E_2 \cup E_3 \cup E_4)$, the assertion follows from Lemma 8.1.4 [c).]. If $P \in \bigcup_{i\in\{2,3,4\},\ j\in\{1,2\}} \left(L_{i,j} \cup L'_{i,j}\right) \backslash (E_2 \cup E_3 \cup E_4)$, the assertion follows from Lemma 8.1.45. Otherwise, the assertion follows from Lemma 8.1.1.

XIII. If $P \in (E_1 \cap E'_1) \cup (E_2 \cap E'_2)$, the assertion follows from Lemma 8.1.15. If $P \in (E_1 \cup E'_1 \cup E_2 \cup E'_2) \backslash \big((E_1 \cap E'_1) \cup (E_2 \cap E'_2)\big)$, the assertion follows from Lemma 8.1.14 [e).]. If $P \in (L_{12} \cup L''_{12}) \backslash (E_1 \cup E'_1 \cup E_2 \cup E'_2)$, the assertion follows from Lemma 8.1.8 [a).]. If $P \in \bigcup_{i,j\in\{1,2\}} \left(L_{i,j} \cup L'_{i,j}\right) \backslash (E_1 \cup E'_1 \cup E_2 \cup E'_2)$, the assertion follows from Lemma 8.1.44. Otherwise, the assertion follows from Lemma 8.1.1.

XIV. If $P \in (E_1 \cap E_1') \cup (E_2 \cap E_2')$, the assertion follows from Lemma 8.1.15. If $P \in (E_1 \cup E_1' \cup E_2 \cup E_2') \backslash \big((E_1 \cap E_1') \cup (E_2 \cap E_2')\big)$, the assertion follows from Lemma 8.1.14 [f).]. If $P \in E_3$, the assertion follows from Lemma 8.1.12. If $P \in (L_{12} \cup L''_{12}) \backslash (E_1 \cup E_1' \cup E_2 \cup E_2')$, the assertion follows from Lemma 8.1.8 [a).]. If $P \in (L_{13} \cup L'_{13} \cup L_{23} \cup L'_{23}) \backslash (E_1 \cup E_1' \cup E_2 \cup E_2' \cup E_3)$, the assertion follows from Lemma 8.1.4 [d).]. If $P \in (L_{1,1} \cup L'_{1,1} \cup L_{2,1} \cup L'_{2,1}) \backslash (E_1 \cup E_1' \cup E_2 \cup E_2')$. Otherwise, the assertion follows from Lemma 8.1.1.

XV. If $P \in (E_1 \cap E_1') \cup (E_2 \cap E_2') \cup (E_3 \cap E_3')$, the assertion follows from Lemma 8.1.15. If $P \in (E_1 \cup E_1' \cup E_2 \cup E_2' \cup E_3 \cup E_3') \backslash \big((E_1 \cap E_1') \cup (E_2 \cap E_2') \cup (E_3 \cap E_3')\big)$, the assertion follows from Lemma 8.1.14 [g).]. If $P \in \bigcup_{i,j \in \{1,2,3\},\, i<j} \big(L_{ij} \cup L'_{ij})\big)$, the assertion follows from Lemma 8.1.8 [a).]. Otherwise, the assertion follows from Lemma 8.1.1.

XVI. If $P \in E_2$, the assertion follows from Lemma 8.1.22 [a).]. If $P \in (E_1 \cup E_1') \backslash E_2$, the assertion follows from Lemma 8.1.16 [a).]. If $P \in (L_{2,1} \cup L_{2,2}) \backslash E_2$, the assertion follows from Lemma 8.1.3 [b).]. If $P \in \bigcup_{i \in \{1,2,3,4\}} \big(L_{1,i} \cup L'_{1,i}\big)$, the assertion follows from Lemma 8.1.43. Otherwise, the assertion follows from Lemma 8.1.1.

XVII. If $P \in E_2$, the assertion follows from Lemma 8.1.22. If $P \in (E_1 \cup E_1') \backslash E_2$, the assertion follows from Lemma 8.1.16 [b).]. If $P \in E_3$, the assertion follows from Lemma 8.1.12. If $P \in (L_{13} \cup L'_{13}) \backslash (E_1 \cup E_1' \cup E_3)$, the assertion follows from Lemma 8.1.6 [a).]. If $P \in (L_{2,1} \cup L_{2,2}) \backslash E_2$, the assertion follows from Lemma 8.1.3 [b).]. If $P \in (L_{3,1} \cup L'_{3,1} \cup L_{3,2} \cup L'_{3,2}) \backslash E_3$, the assertion follows from Lemma 8.1.45. If $P \in (L_{1,1} \cup L'_{1,1} \cup L_{1,2} \cup L'_{1,2}) \backslash (E_1 \cup E_1')$, the assertion follows from Lemma 8.1.43. Otherwise, the assertion follows from Lemma 8.1.1.

XVIII. If $P \in E_2$, the assertion follows from Lemma 8.1.22 [b).]. If $P \in (E_1 \cup E_1') \backslash E_2$, the assertion follows from Lemma 8.1.16 [a).]. If $P \in E_3$, the assertion follows from Lemma 8.1.12. If $P \in L_{23} \backslash (E_2 \cup E_3)$, the assertion follows from Lemma 8.1.13 [b).]. If $P \in \bigcup_{i \in \{1,2,3\}} \big(L_{3,i} \cup L'_{3,i}\big) \backslash E_3$, the assertion follows from Lemma 8.1.45. If $P \in \bigcup_{j \in \{1,2,3,4\}} \big(L_{1,j} \cup L'_{1,j}\big) \backslash E_3$, the assertion follows from Lemma 8.1.43. Otherwise, the assertion follows from Lemma 8.1.1.

XIX. If $P \in E_2$, the assertion follows from Lemma 8.1.22 [a).]. If $P \in (E_1 \cup E_1') \backslash E_2$, the assertion follows from Lemma 8.1.16 [c).]. If $P \in E_3 \cup E_4$, the assertion follows from Lemma 8.1.12. If $P \in (L_{13} \cup L'_{13} \cup L_{14} \cup L'_{14}) \backslash (E_1 \cup E_1' \cup E_3 \cup E_4)$, the assertion follows from Lemma 8.1.6 [b).]. If $P \in (L_{2,1} \cup L_{2,2}) \backslash E_2$, the assertion follows from Lemma 8.1.3 [b).]. If $P \in (L_{34} \cup L_{34}) \backslash (E_3 \cup E_4)$, the assertion follows from Lemma 8.1.3 [a).]. Otherwise, the assertion follows from Lemma 8.1.1.

XX. If $P \in E_2$, the assertion follows from Lemma 8.1.22. If $P \in (E_1 \cup E_1') \backslash E_2$, the assertion follows from Lemma 8.1.16 [b).]. If $P \in E_3 \cup E_4$, the assertion follows from Lemma 8.1.12. If $P \in L_{23} \backslash (E_2 \cup E_3)$, the assertion follows from Lemma 8.1.13 [b).]. If $P \in (L_{14} \cup L'_{14}) \backslash (E_1 \cup E_1' \cup E_4)$, the assertion follows from Lemma 8.1.6 [b).]. If $P \in (L_{34} \cup$

$L'_{34})\backslash(E_3 \cup E_4)$, the assertion follows from Lemma 8.1.3 [a).]. If $P \in (L_{3,1} \cup L_{3,1})\backslash E_3$, the assertion follows from Lemma 8.1.45. If $P \in \bigcup_{k\in\{1,2\}} \left(L_{1,k} \cup L'_{1,k}\right)\backslash(E_1 \cup E'_1)$, the assertion follows from Lemma 8.1.43. Otherwise, the assertion follows from Lemma 8.1.1.

XXI. If $P \in E_2$, the assertion follows from Lemma 8.1.22 [b).]. If $P \in (E_1 \cup E'_1)\backslash E_2$, the assertion follows from Lemma 8.1.16 [c).]. If $P \in E_3 \cup E_4 \cup E_5$, the assertion follows from Lemma 8.1.12. If $P \in L_{345}\backslash(E_3 \cup E_4 \cup E_5)$, the assertion follows from Lemma 8.1.13 [a).]. If $P \in L_{25}\backslash(E_2 \cup E_5)$, the assertion follows from Lemma 8.1.13 [b).]. If $P \in (L_{13} \cup L'_{13} \cup L_{14} \cup L'_{14})\backslash(E_1 \cup E'_1 \cup E_3 \cup E_4)$, the assertion follows from Lemma 8.1.6 [b).]. If $P \in (L_{5,1} \cup L'_{5,1})\backslash E_5$, the assertion follows from Lemma 8.1.45. Otherwise, the assertion follows from Lemma 8.1.1.

XXII. If $P \in E_2$, the assertion follows from Lemma 8.1.22 [a).]. If $P \in (E_1 \cup E'_1)\backslash E_2$, the assertion follows from Lemma 8.1.16 [d).]. If $P = E_3 \cap E'_3$, the assertion follows from Lemma 8.1.15. If $P \in (E_3 \cup E'_3)\backslash(E_3 \cap E'_3)$, the assertion follows from Lemma 8.1.14 [h).]. If $P \in (L_{13} \cup L''_{13})\backslash(E_1 \cup E'_1 \cup E_3 \cup E'_3)$, the assertion follows from Lemma 8.1.9. If $P \in (L_{2,1} \cup L_{2,2})\backslash E_2$, the assertion follows from Lemma 8.1.3 [b).]. If $P \in (L_{3,1} \cup L'_{3,1} \cup L_{3,2} \cup L'_{3,2})\backslash(E_3 \cup E'_3)$. If $P \in (L_{1,1} \cup L'_{1,1})\backslash(E_1 \cup E'_1)$, the assertion follows from Lemma 8.1.43. Otherwise, the assertion follows from Lemma 8.1.1.

XXIII. If $P \in E_2$, the assertion follows from Lemma 8.1.22 [b).]. If $P \in (E_1 \cup E'_1)\backslash E_2$, the assertion follows from Lemma 8.1.16 [d).]. If $P = E_3 \cap E'_3$, the assertion follows from Lemma 8.1.15. If $P \in (E_3 \cup E'_3)\backslash(E_3 \cap E'_3)$, the assertion follows from Lemma 8.1.14 [i).]. If $P \in (L_{13} \cup L''_{13})\backslash(E_1 \cup E'_1 \cup E_3 \cup E'_3)$. If $P \in E_4$, the assertion follows from Lemma 8.1.12. If $P \in L_{24}\backslash(E_2 \cup E_4)$, the assertion follows from Lemma 8.1.13 [b).]. If $P \in (L_{34} \cup L'_{34})\backslash(E_3 \cup E'_3 \cup E_4)$, the assertion follows from Lemma 8.1.4 [e).]. If $P \in (L_{1,1} \cup L'_{1,1})\backslash(E_1 \cup E'_1)$, the assertion follows from Lemma 8.1.43. Otherwise, the assertion follows from Lemma 8.1.1.

XXIV. If $P \in E_2 \cup E_4$, the assertion follows from Lemma 8.1.22 [a).]. If $P \in (E_1 \cup E'_1 \cup E_3 \cup E'_3)\backslash(E_2 \cup E_4)$, the assertion follows from Lemma 8.1.16 [e).]. If $P \in (L_{13} \cup L''_{13})\backslash(E_1 \cup E'_1 \cup E_3 \cup E'_3)$, the assertion follows from Lemma 8.1.13 [c).]. If $P \in (L_{2,1} \cup L_{2,2} \cup L_{4,1} \cup L_{4,2})\backslash(E_2 \cup E_4)$, the assertion follows from Lemma 8.1.3. Otherwise, the assertion follows from Lemma 8.1.1.

XXV. If $P \in_2 \cup E_4$, the assertion follows from Lemma 8.1.22 [b).]. If $P \in (E_1 \cup E'_1 \cup E_3 \cup E'_3)\backslash(E_2 \cup E_4)$, the assertion follows from Lemma 8.1.16 [e).]. If $P \in E_5$, the assertion follows from Lemma 8.1.12. If $P \in (L_{25} \cup L_{45})\backslash(E_1 \cup E'_1 \cup E_3 \cup E'_3 \cup E_5)$, the assertion follows from Lemma 8.1.13 [b).]. If $P \in (L_{13} \cup L''_{13} \cup E_5)\backslash(E_1 \cup E'_1 \cup E_3 \cup E'_3)$, the assertion follows from Lemma 8.1.13 [c).]. Otherwise, the assertion follows from Lemma 8.1.1.

XXVI. If $P \in E_2 \cup E_2'$, the assertion follows from Lemma 8.1.23. If $P \in (E_1 \cup E_1') \backslash (E_2 \cup E_2')$, the assertion follows from Lemma 8.1.17 [a).]. If $P \in (L_{2,1} \cup L_{2,1}') \backslash (E_2 \cup E_2')$, the assertion follows from Lemma 8.1.5 [a).]. If $P \in \bigcup_{i \in \{1,2,3\}} \left(L_{1,i} \cup L_{1,i}'\right)$, the assertion follows from Lemma 8.1.42. Otherwise, the assertion follows from Lemma 8.1.1.

XXVII. If $P \in E_2 \cup E_2'$, the assertion follows from Lemma 8.1.23. If $P \in (E_1 \cup E_1') \backslash (E_2 \cup E_2')$, the assertion follows from Lemma 8.1.17 [b).]. If $P \in E_3$, the assertion follows from Lemma 8.1.12. If $P \in (L_{2,1} \cup L_{2,1}') \backslash (E_2 \cup E_2')$, the assertion follows from Lemma 8.1.5 [b).]. If $P \in (L_{13} \cup L_{13}') \backslash (E_1 \cup E_1' \cup E_3)$, the assertion follows from Lemma 8.1.7. If $P \in (L_{3,1} \cup L_{3,1}') \backslash E_3$, the assertion follows from Lemma 8.1.45. If $P \in (L_{1,1} \cup L_{1,1}') \backslash (E_1 \cup E_1')$, the assertion follows from Lemma 8.1.42. Otherwise, the assertion follows from Lemma 8.1.1.

XXVIII. If $P \in E_2 \cup E_2'$, the assertion follows from Lemma 8.1.23. If $P \in (E_1 \cup E_1') \backslash (E_2 \cup E_2')$, the assertion follows from Lemma 8.1.17 [c).]. If $P = E_3 \cap E_3'$, the assertion follows from Lemma 8.1.15. If $P \in (E_3 \cup E_3) \backslash (E_3 \cap E_3)$, the assertion follows from Lemma 8.1.14 [j).]. If $P \in (L_{13} \cup L_{13}'') \backslash (E_1 \cup E_1' \cup E_3 \cup E_3)$, the assertion follows from Lemma 8.1.11. If $P \in (L_{2,1} \cup L_{2,1}') \backslash (E_2 \cup E_2')$, the assertion follows from Lemma 8.1.5 [c).]. If $P \in (L_{3,1} \cup L_{3,1}') \backslash (E_3 \cup E_3')$. Otherwise, the assertion follows from Lemma 8.1.1.

XXIX. If $P \in E_3$, the assertion follows from Lemma 8.1.26 [a).]. If $P \in (E_2 \cup E_2') \backslash E_3$, the assertion follows from Lemma 8.1.27 [a).]. If $P \in (E_1 \cup E_1') \backslash (E_2 \cup E_2')$, the assertion follows from Lemma 8.1.18 [a).].. If $P \in (L_{2,1} \cup L_{2,1}') \backslash (E_2 \cup E_2')$, the assertion follows from Lemma 8.1.8 [b).]. If $P \in \bigcup_{i \in \{1,2\}} \left(L_{1,i} \cup L_{1,i}'\right) \backslash (E_1 \cup E_1')$, the assertion follows from Lemma 8.1.41. Otherwise, the assertion follows from Lemma 8.1.1.

XXX. If $P \in E_3$, the assertion follows from Lemma 8.1.30 [a).]. If $P \in (E_2 \cup E_2') \backslash E_3$, the assertion follows from Lemma 8.1.24. If $P \in (E_1 \cup E_1') \backslash (E_2 \cup E_2')$, the assertion follows from Lemma 8.1.20 [a).]. If $P \in L_{3,1} \backslash E_3$, the assertion follows from Lemma 8.1.13 [d).]. If $P \in \bigcup_{i \in \{1,2,3\}} \left(L_{1,i} \cup L_{1,i}'\right)$, the assertion follows from Lemma 8.1.41. Otherwise, the assertion follows from Lemma 8.1.1.

XXXI. If $P \in E_3$, the assertion follows from Lemma 8.1.26 [a).]. If $P \in (E_2 \cup E_2') \backslash E_3$, the assertion follows from Lemma 8.1.27 [a).]. If $P \in (E_1 \cup E_1') \backslash (E_2 \cup E_2')$, the assertion follows from Lemma 8.1.18 [b).]. If $P \in E_4$, the assertion follows from Lemma 8.1.12. If $P \in (L_{2,1} \cup L_{2,1}') \backslash (E_2 \cup E_2')$, the assertion follows from Lemma 8.1.8 [b).]. If $P \in (L_{14} \cup L_{14}') \backslash (E_1 \cup E_1' \cup E_4)$, the assertion follows from Lemma 8.1.8 [c).]. Otherwise, the assertion follows from Lemma 8.1.1.

XXXII. If $P \in E_3$, the assertion follows from Lemma 8.1.30 [a).]. If $P \in (E_2 \cup E_2') \backslash E_3$, the assertion follows from Lemma 8.1.24. If $P \in (E_1 \cup E_1') \backslash (E_2 \cup E_2')$, the assertion follows from Lemma 8.1.20 [b).]. If $P \in E_4$, the assertion follows from Lemma 8.1.12. If $P \in L_{3,1} \backslash E_3$, the assertion follows from Lemma 8.1.13 [d).]. If $P \in (L_{14} \cup L_{14}') \backslash (E_1 \cup E_1' \cup E_4)$, the assertion follows from Lemma 8.1.2. If $P \in (L_{1,1} \cup L_{1,1}) \backslash (E_1 \cup E_1')$, the assertion follows from Lemma 8.1.41. Otherwise, the assertion follows from Lemma 8.1.1.

XXXIII. If $P \in E_3$, the assertion follows from Lemma 8.1.30 [a).]. If $P \in (E_2 \cup E_2')\backslash E_3$, the assertion follows from Lemma 8.1.24. If $P \in (E_1 \cup E_1')\backslash(E_2 \cup E_2')$, the assertion follows from Lemma 8.1.20 [c).]. If $P = E_4 \cap E_4$, the assertion follows from Lemma 8.1.15 If $P \in (E_4 \cup E_4')\backslash(E_4 \cap E_4')$, the assertion follows from Lemma 8.1.14 [k).]. If $P \in L_{3,1}\backslash E_3$, the assertion follows from Lemma 8.1.13 [d).]. If $P \in (L_{14} \cup L_{14}')\backslash(E_1 \cup E_1' \cup E_4 \cup E_4')$, the assertion follows from Lemma 8.1.13 [e).]. Otherwise, the assertion follows from Lemma 8.1.1.

XXXIV. If $P \in E_3 \cup E_3'$, the assertion follows from Lemma 8.1.28. If $P \in (E_2 \cup E_2')\backslash(E_3 \cup E_3')$, the assertion follows from Lemma 8.1.29. If $P \in (E_1 \cup E_1')\backslash(E_2 \cup E_2')$, the assertion follows from Lemma 8.1.19. If $P \in (L_{2,1} \cup L_{2,1}')\backslash(E_2 \cup E_2')$, the assertion follows from Lemma 8.1.10. If $P \in (L_{1,1} \cup L_{1,1}')\backslash(E_1 \cup E_1')$, the assertion follows from Lemma 8.1.40. Otherwise, the assertion follows from Lemma 8.1.1.

XXXV. If $P \in E_4$, the assertion follows from Lemma 8.1.31 [a).]. If $P \in (E_3 \cup E_3')\backslash E_4$, the assertion follows from Lemma 8.1.32. If $P \in (E_2 \cup E_2')\backslash(E_3 \cup E_3')$. By Lemma 8.1.30 [b).]. If $P \in (E_1 \cup E_1')\backslash(E_2 \cup E_2')$. By Lemma 8.1.21. If $P \in (L_{2,1} \cup L_{2,1}')\backslash(E_2 \cup E_2')$, the assertion follows from Lemma 8.1.13 [f).]. Otherwise, the assertion follows from Lemma 8.1.1.

XXXVI. If $P \in E$, the assertion follows from Lemma 8.1.31 [b).]. If $P \in (E_1 \cup E_2 \cup E_3)\backslash E$, the assertion follows from Lemma 8.1.22 [c).]. If $P \in \bigcup_{i\in\{1,2,3\},j\in\{1,2\}} L_{i,j}\backslash(E_1 \cup E_2 \cup E_3)$, the assertion follows from Lemma 8.1.3 [c).]. Otherwise, the assertion follows from Lemma 8.1.1.

XXXVII. If $P \in E$, the assertion follows from Lemma 8.1.31 [b).]. If $P \in (E_1 \cup E_1')\backslash E$, the assertion follows from Lemma 8.1.22 [c).]. If $P \in E_2\backslash E$, the assertion follows from Lemma 8.1.22 [d).]. If $P \in E_3$, the assertion follows from Lemma 8.1.12. If $P \in L_{23}\backslash(E_2 \cup E_3)$, the assertion follows from Lemma 8.1.13 [f).]. If $P \in \bigcup_{i\in\{1,2\}} (L_{1,i} \cup L_{1,i}')\backslash(E_1 \cup E_1')$, the assertion follows from Lemma 8.1.3 [c).]. If $P \in (L_{3,1} \cup L_{3,1}' \cup L_{3,2} \cup L_{3,2}')\backslash E_3$, the assertion follows from Lemma 8.1.45. Otherwise, the assertion follows from Lemma 8.1.1.

XXXVIII. If $P \in E$, the assertion follows from Lemma 8.1.31 [b).]. If $P \in E_2\backslash E$, the assertion follows from Lemma 8.1.22 [c).]. If $P \in (E_1 \cup E_1')\backslash E$, the assertion follows from Lemma 8.1.22 [d).]. If $P \in E_3 \cup E_3'$, the assertion follows from Lemma 8.1.12. If $P \in (L_{13} \cup L_{13}'')\backslash(E_1 \cup E_1' \cup E_3 \cup E_3')$, the assertion follows from Lemma 8.1.13 [f).]. If $P \in (L_2 \cup L_2')\backslash E_2$, the assertion follows from Lemma 8.1.3 [c).]. If $P \in (L_3 \cup L_3')\backslash(E_3 \cup E_3')$, the assertion follows from Lemma 8.1.3 [a).]. Otherwise, the assertion follows from Lemma 8.1.1.

XXXIX. If $P \in E$, the assertion follows from Lemma 8.1.31 [b).]. If $P \in (E_1 \cup E_2 \cup E_3)\backslash E$, the assertion follows from Lemma 8.1.22 [d).]. If $P \in E_1' \cup E_2' \cup E_3'$, the assertion follows from Lemma 8.1.12. If $P \in (L_1 \cup L_2 \cup L_3)\backslash(E_1 \cup E_2 \cup E_3 \cup E_1' \cup E_2' \cup E_3')$, the assertion follows from Lemma 8.1.13 [f).]. If $P \in L_{123}\backslash(E_1' \cup E_2' \cup E_3')$, the assertion follows from Lemma 8.1.13 [a).]. Otherwise, the assertion follows from Lemma 8.1.1.

XL. If $P \in E$, the assertion follows from Lemma 8.1.33 [a).]. If $P \in E_2 \backslash E$, the assertion follows from Lemma 8.1.31 [c).]. If $P \in (E_1 \cup E_1') \backslash E$, the assertion follows from Lemma 8.1.25. If $P \in E_3 \backslash E_2$, the assertion follows from Lemma 8.1.22 [e).]. If $P \in (L_1 \cup L_1') \backslash (E_1 \cup E_1')$, the assertion follows from Lemma 8.1.6 [c).]. If $P \in (L_{3,1} \cup L_{3,2}) \backslash E_3$, the assertion follows from Lemma 8.1.3 [e).]. Otherwise, the assertion follows from Lemma 8.1.1.

XLI. If $P \in E$, the assertion follows from Lemma 8.1.33 [a).]. If $P \in E_2 \backslash E$, the assertion follows from Lemma 8.1.31 [c).]. If $P \in (E_1 \cup E_1') \backslash E$, the assertion follows from Lemma 8.1.25. If $P \in E_3 \backslash E_2$, the assertion follows from Lemma 8.1.22 [f).]. If $P \in E_4$, the assertion follows from Lemma 8.1.12. If $P \in L_{34} \backslash (E_3 \cup E_4)$, the assertion follows from Lemma 8.1.13 [g).]. If $P \in (L_1 \cup L_1') \backslash (E_1 \cup E_1')$, the assertion follows from Lemma 8.1.6 [d).]. If $P \in (L_{4,1} \cup L_{4,1}') \backslash E_4$, the assertion follows from Lemma 8.1.45. Otherwise, the assertion follows from Lemma 8.1.1.

XLII. If $P \in E_2$, the assertion follows from Lemma 8.1.36. If $P \in E_3 \backslash E_2$, the assertion follows from Lemma 8.1.33 [b).]. If $P \in E \backslash E_2$, the assertion follows from Lemma 8.1.30 [c).]. If $P \in E_4 \backslash E_3$, the assertion follows from Lemma 8.1.31 [d).]. If $P \in E_1 \backslash E_2$, the assertion follows from Lemma 8.1.26 [b).]. If $P \in L \backslash E$, the assertion follows from Lemma 8.1.13 [h).]. If $P \in E_5 \backslash E_4$, the assertion follows from Lemma 8.1.22 [g).]. If $P \in (L_5 \cup L_5') \backslash E_5$, the assertion follows from Lemma 8.1.3 [e).]. Otherwise, the assertion follows from Lemma 8.1.1.

XLIII. If $P \in E$, the assertion follows from Lemma 8.1.36 [a).]. If $P \in E_4 \backslash E_3$, the assertion follows from Lemma 8.1.33 [b).]. If $P \in E \backslash E_2$, the assertion follows from Lemma 8.1.30. If $P \in E_4 \backslash E_2$, the assertion follows from Lemma 8.1.31. If $P \in E_1 \backslash E_2$, the assertion follows from Lemma 8.1.26 [b).]. If $P \in L \backslash E$, the assertion follows from Lemma 8.1.13 [h).]. If $P \in E_5 \backslash E_4$, the assertion follows from Lemma 8.1.22 [h).]. If $P \in E_6$, the assertion follows from Lemma 8.1.12. If $P \in L_{56} \backslash (E_5 \cup E_6)$, the assertion follows from Lemma 8.1.13 [i).]. Otherwise, the assertion follows from Lemma 8.1.1.

XLIV. If $P \in E$, the assertion follows from Lemma 8.1.37 [a).]. If $P \in E \backslash E_3$, the assertion follows from Lemma 8.1.31 [e).]. If $P \in (E_2 \cup E_2') \backslash E_3$, the assertion follows from Lemma 8.1.34. If $P \in (E_1 \cup E_1') \backslash (E_2 \cup E_2')$, the assertion follows from Lemma 8.1.27 [b).]. If $P \in (L_1 \cup L_1') \backslash (E_1 \cup E_1')$, the assertion follows from Lemma 8.1.8 [d).]. Otherwise, the assertion follows from Lemma 8.1.1.

XLV. If $P \in E_3$, the assertion follows from Lemma 8.1.39. If $P \in E_4 \backslash E_3$, the assertion follows from Lemma 8.1.38. If $P \in E \backslash E_3$, the assertion follows from Lemma 8.1.35. If $P \in E_2 \backslash E_3$, the assertion follows from Lemma 8.1.37 [b).]. If $P \in E_5 \backslash E_4$, the assertion follows from Lemma 8.1.36 [b).]. If $P \in E_1 \backslash E_2$, the assertion follows from Lemma 8.1.31 [f).]. If $P \in E_6 \backslash E_5$, the assertion follows from Lemma 8.1.30 [d).]. If $P \in L_6 \backslash E_6$, the assertion follows from Lemma 8.1.13 [j).]. Otherwise, the assertion follows from Lemma 8.1.1.

□

Chapter 9

Du Val del Pezzo Surfaces of Degree 1

In (Araujo et al., 2023, Lemma 2.16) it was proven that $\delta(X) = \frac{15}{7}$ when X is a smooth del Pezzo surface of degree 1 and $|-K_X|$ contains a cuspidal curve, and $\delta(X) = \frac{12}{5}$ when X is a smooth del Pezzo surface of degree 1 and $|-K_X|$ does not contain a cuspidal curve.

We consider a Del Pezzo surface X of degree one with at worst Du Val singularities and denote its minimal resolution by $\pi : S \to X$. The surface X can be embedded as a degree six hypersurface in the weighted projective space $\mathbb{P}(1,1,2,3)$, given by the equation

$$w^2 = az^3 + z^2 f_2(x,y) + zf_4(x,y) + f_6(x,y),$$

where f_2, f_4, f_6 are homogeneous polynomials in X and y of degrees 2, 4, and 6 respectively, and $a \in \mathbb{C}$ is a constant. This defines X as a double cover $\varphi : X \to \mathbb{P}(1,1,2)$, given by:

$$(x : y : z : w) \mapsto (x : y : z),$$

branched along the sextic curve

$$R : az^3 + z^2 f_2(x,y) + zf_4(x,y) + f_6(x,y) = 0 \subset \mathbb{P}(1,1,2).$$

The branch curve R has degree six an is in general singular. There is a natural one-to-one correspondence between the singularities of R and the singular points of the surface X; that is, the singularities of X lie precisely above the singular points of R. As shown in Kosta (2009), the singular points of X are not contained in the base locus of the anti-canonical linear system $|-K_X|$. In other words, they are not fixed points of this system.

In this section, we compute δ-invariants of Du Val del Pezzo surfaces of degree 1.

MAIN THEOREM Let X be a Du Val del Pezzo surface of degree 1. Then X can be realized as the double cover $X \xrightarrow{2:1} \mathbb{P}(1,1,2)$, which is ramified along a sextic curve $R \in \mathbb{P}(1,1,2)$. Then the δ-invariant of X is uniquely determined by the type of singularities on X and unique element $\mathscr{C}$ of $|-K_X|$ containing each of singular points which is given in the following table:

Type of singularity	$\delta(X)$
$\mathbb{A}_1$, $2\mathbb{A}_1$, $3\mathbb{A}_1$, $4\mathbb{A}_1$, $5\mathbb{A}_1$, $6\mathbb{A}_1$, $7\mathbb{A}_1$, $8\mathbb{A}_1$ all elements of $\lvert -K_X\rvert$ containing singular points are nodal	2
$\mathbb{A}_1$, $2\mathbb{A}_1$, $3\mathbb{A}_1$, $4\mathbb{A}_1$, $5\mathbb{A}_1$, $6\mathbb{A}_1$, $7\mathbb{A}_1$, $8\mathbb{A}_1$ some elements of $\lvert -K_X\rvert$ containing singular points are cuspidal	$\frac{9}{5}$
$\mathbb{A}_2$, $\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_2+2\mathbb{A}_1$, $\mathbb{A}_2+3\mathbb{A}_1$, $\mathbb{A}_2+4\mathbb{A}_1$, $2\mathbb{A}_2$, $2\mathbb{A}_2+\mathbb{A}_1$, $2\mathbb{A}_2+2\mathbb{A}_1$, $3\mathbb{A}_2$, $3\mathbb{A}_2+\mathbb{A}_1$, $4\mathbb{A}_2$ all elements of $\lvert -K_X\rvert$ containing $\mathbb{A}_2$ singular points are nodal	$\frac{12}{7}$
$\mathbb{A}_2$, $\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_2+2\mathbb{A}_1$, $\mathbb{A}_2+3\mathbb{A}_1$, $\mathbb{A}_2+4\mathbb{A}_1$, $2\mathbb{A}_2$, $2\mathbb{A}_2+\mathbb{A}_1$, $2\mathbb{A}_2+2\mathbb{A}_1$, $3\mathbb{A}_2$, $3\mathbb{A}_2+\mathbb{A}_1$, $4\mathbb{A}_2$ some elements of $\lvert -K_X\rvert$ containing $\mathbb{A}_2$ singular points are cuspidal	$\frac{3}{2}$
$\mathbb{A}_3$, $\mathbb{A}_3+\mathbb{A}_1$, $\mathbb{A}_3+2\mathbb{A}_1$, $\mathbb{A}_3+3\mathbb{A}_1$, $\mathbb{A}_3+4\mathbb{A}_1$, $\mathbb{A}_3+\mathbb{A}_2$, $\mathbb{A}_3+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_3+\mathbb{A}_2+2\mathbb{A}_1$, $2\mathbb{A}_3$, $2\mathbb{A}_3+\mathbb{A}_1$, $2\mathbb{A}_3+2\mathbb{A}_1$	$\frac{3}{2}$
$\mathbb{A}_4$, $\mathbb{A}_4+\mathbb{A}_1$, $\mathbb{A}_4+2\mathbb{A}_1$, $\mathbb{A}_4+\mathbb{A}_2$, $\mathbb{A}_4+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_4+\mathbb{A}_3$, $2\mathbb{A}_4$	$\frac{4}{3}$
$\mathbb{A}_5$, $\mathbb{A}_5+\mathbb{A}_1$, $\mathbb{A}_5+2\mathbb{A}_1$, $\mathbb{A}_5+\mathbb{A}_2$, $\mathbb{A}_5+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_5+\mathbb{A}_3$	$\frac{6}{5}$
$\mathbb{A}_6$, $\mathbb{A}_6+\mathbb{A}_1$	$\frac{9}{8}$
$\mathbb{A}_7$, $\mathbb{A}_7+\mathbb{A}_1$ and R irreducible	$\frac{18}{17}$
$\mathbb{A}_7$, $\mathbb{A}_7+\mathbb{A}_1$ and R reducible	1
$\mathbb{A}_8$, $\mathbb{D}_4$, $\mathbb{D}_4+\mathbb{A}_1$, $\mathbb{D}_4+2\mathbb{A}_1$, $\mathbb{D}_4+3\mathbb{A}_1$, $\mathbb{D}_4+4\mathbb{A}_1$,$\mathbb{D}_4+\mathbb{A}_2$, $\mathbb{D}_4+\mathbb{A}_3$, $2\mathbb{D}_4$	1
$\mathbb{D}_5$, $\mathbb{D}_5+\mathbb{A}_1$, $\mathbb{D}_5+2\mathbb{A}_1$, $\mathbb{D}_5+\mathbb{A}_2$, $\mathbb{D}_5+\mathbb{A}_3$	$\frac{6}{7}$
$\mathbb{D}_6$, $\mathbb{D}_6+\mathbb{A}_1$, $\mathbb{D}_6+2\mathbb{A}_1$	$\frac{3}{4}$
$\mathbb{D}_7$	$\frac{2}{3}$
$\mathbb{D}_8$, $\mathbb{E}_6$, $\mathbb{E}_6+\mathbb{A}_1$, $\mathbb{E}_6+\mathbb{A}_2$	$\frac{3}{5}$
$\mathbb{E}_7$, $\mathbb{E}_7+\mathbb{A}_1$	$\frac{3}{7}$
$\mathbb{E}_8$	$\frac{3}{11}$

Table 9.1: δ-invariants of Du Val del Pezzo surfaces of degree 1

Note that when X has $\mathbb{A}_7$ singularities δ-invariant depends on whether R is reducible or irreducible.

To understand the anti-canonical system on the smooth surface S, we apply the Riemann–Roch theorem together with Serre duality and the Kawamata–Viehweg vanishing theorem. For the divisor $-K_S$, we have

$$\chi(\mathscr{O}_S(-K_S)) = h^0(S,\mathscr{O}_S(-K_S)) - h^1(S,\mathscr{O}_S(-K_S)) + h^2(S,\mathscr{O}_S(-K_S)).$$

Since $-K_S$ is nef and big, the vanishing theorems imply $h^1 = h^2 = 0$, and therefore

$$h^0(S,\mathscr{O}_S(-K_S)) = \chi(\mathscr{O}_S(-K_S)) = \frac{1}{2}K_S^2 + 1 = K_S^2 + 1.$$

Thus, the anti-canonical system $|-K_S|$ has dimension

$$\dim|-K_S| = h^0(S, \mathscr{O}_S(-K_S)) - 1 = K_S^2 = 1,$$

DEFINITION. Let $\pi : S \to X$ be a resolution of a point P on a normal surface X, and let $E = \sum E_i$ denote the exceptional divisor over P. Then there exists a unique effective exceptional divisor $\Gamma = \sum a_i E_i, \quad a_i \in \mathbb{Z}_{>0}$, satisfying the following properties:

1. $\Gamma > 0$,
2. $\Gamma \cdot E_i \leq 0$ for every component E_i,
3. Γ is minimal with respect to this property.

The divisor Γ is called the **fundamental cycle** of the configuration $\{E_i\}$.

In the context of Del Pezzo surfaces of degree one, Kosta (2009) shows the following result: let $H \in |-K_S|$ be an anti-canonical divisor on the resolution S, and let Γ be the fundamental cycle of the exceptional divisor over a Du Val singularity. If the curve H contains a point of Γ, then $H = C + \Gamma$, where $C \subset S$ is the strict transform of a (-1)-curve $\mathscr{C}$ on X. Moreover, all fundamental cycles arising from Du Val singularities on degree one Del Pezzo surfaces are explicitly described in Kosta (2009), including their configurations and intersection properties. Let $C \subset S$ be a (-1)-curve that arises as the strict transform of a curve $\mathscr{C} \subset X$. Contracting C yields a weak resolution of a Du Val Del Pezzo surface of degree two. In the preceding section, we provided a complete classification of the dual graphs formed by (-1)- and (-2)-curves on such surfaces. It is important to note that all (-1)-curves on S intersecting the exceptional divisors are strict transforms of (-1)-curves on weak Del Pezzo surfaces of degree two. Throughout this section, we make systematic use of the classification of all (-1)-curves that meet the exceptional divisors, as described in the previous chapter.

9.1 Finding δ-invariants for degree 1

$\mathbb{A}_1$ singularity on Du Val Del Pezzo surfaces of degree 1 such that $\mathscr{C}$ is nodal

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_1$ singularity at point $\mathscr{P}$. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$ and it has a node in $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X) = 2$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathscr{C}$ on S and E is the exceptional divisor. We have $-K_S \sim C + E$. Let P be a point on S.

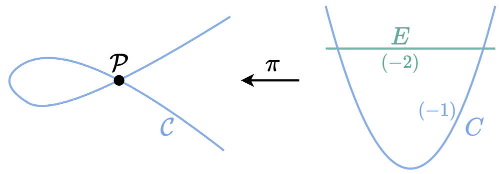


Figure 9.1: Picture: $(-K_S)^2 = 1$, $\mathbb{A}_1$ singularity (nodal)

Suppose $P \in E$. Then $\tau(E) = 1$ and the Zariski decomposition of the divisor $-K_S - vE \sim C + (1-v)E$ is given by:

$$P(v) = \begin{cases} -K_S - vE \text{ if } v \in \left[0, \frac{1}{2}\right], \\ -K_S - vE - (2v-1)C \text{ if } v \in \left[\frac{1}{2}, 1\right]. \end{cases} \qquad N(v) = \begin{cases} 0 \text{ if } v \in \left[0, \frac{1}{2}\right], \\ (2v-1)C \text{ if } v \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - 2v^2 \text{ if } v \in \left[0, \frac{1}{2}\right], \\ 2(v-1)^2 \text{ if } v \in \left[\frac{1}{2}, 1\right]. \end{cases} \qquad P(v) \cdot E = \begin{cases} 2v \text{ if } v \in \left[0, \frac{1}{2}\right], \\ 2(1-v) \text{ if } v \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We have $S_S(E) = \frac{1}{2}$. Thus, $\delta_P(S) \leq 2$ for $P \in E$. Moreover, if $P \in E$:

$$h(v) = \begin{cases} 2v^2 \text{ if } v \in \left[0, \frac{1}{2}\right], \\ 2v(1-v) \text{ if } v \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Thus, $S(W^E_{\bullet,\bullet}; P) \leq \frac{1}{2}$ and We get $\delta_P(S) = 2$ for $P \in E$. Which gives us $\delta_{\mathscr{P}}(X) = 2$. □

$\mathbb{A}_1$ singularity on Du Val Del Pezzo surfaces of degree 1 such that $\mathscr{C}$ is cuspidal

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_1$ singularity at point $\mathscr{P}$. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$ and it has a cusp in $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X) = \frac{9}{5}$.

Proof. Consider the blowup $\pi_1 : S_1 \to X$ of X at $\mathscr{P}$ with the exceptional divisor E_1^1 and C^1 is a strict transform of $\mathscr{C}$. Let $\pi_2 \colon S_2 \to S_1$ be the blow up of the point $C^1 \cap E_1^1$ with the exceptional divisor E_2^2 and E_1^2, C^2 are a strict transforms of E_1^1, C^1 respectively. Let $\pi_3 \colon S_3 \to S_2$ be the blow up of the point $C^2 \cap E_1^2 \cap E_2^2$ with the exceptional divisor E and E_1^3, E_2^3, C^3 are a strict transforms of E_1^2, E_2^2, C^2 respectively. Then $(\pi_1 \circ \pi_2 \circ \pi_3)^*(-K_X) \sim C^3 + E_1^3 + 2E_2^3 + 4E$. Let $\theta \colon S_3 \to \overline{S}$ be the contraction of the curves E_1^3 and E_2^3, let $\overline{C} = \theta(C^3)$ and $\overline{E} = \theta(E)$.

Then $\overline{P}_2 = \theta(E_2^3)$ is a quotient singular point of type $\frac{1}{2}(1,1)$ and $\overline{P}_1 = \theta(E_1^3)$ is a quotient singular point of type $\frac{1}{4}(1,1)$ and the intersections are given by:

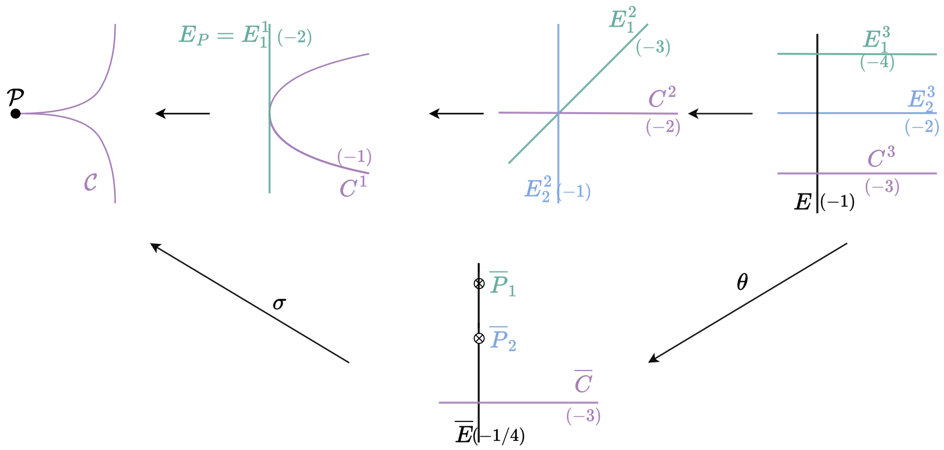


Figure 9.2: Picture: $(-K_S)^2 = 1$, $\mathbb{A}_1$ singularity (cuspidal)

	$\overline{C}$	$\overline{E}$
$\overline{C}$	-3	1
$\overline{E}$	1	$-\frac{1}{4}$

Observe that $-K_{\overline{S}}$ is big. Then $\tau(\overline{E}) = 4$ and the Zariski decomposition of the divisor $\sigma^*(-K_X) - v\overline{E} \sim (4-v)\overline{E} + \overline{C}$ is given by

$$P(v) = \begin{cases} (4-v)\overline{E} + \overline{C} \text{ if } v \in [0,1], \\ (4-v)\overline{E} + \frac{4-v}{3}\overline{C} \text{ if } v \in [1,4]. \end{cases} \qquad N(v) = \begin{cases} 0 \text{ if } v \in [0,1], \\ \frac{v-1}{3}\overline{C} \text{ if } v \in [1,4]. \end{cases}$$

Moreover

$$P(v)^2 = \begin{cases} \frac{(2-v)(2+v)}{4} \text{ if } v \in [0,1], \\ \frac{(4-v)^2}{12} \text{ if } v \in [1,4]. \end{cases} \quad P(v) \cdot \overline{E} = \begin{cases} \frac{v}{4} \text{ if } v \in [0,1], \\ \frac{4-v}{12} \text{ if } v \in [1,4]. \end{cases}$$

So we have $S_S(\overline{E}) = \frac{5}{3}$ for $P \in \overline{E}$. Thus, Thus, $\delta_P(S) \leq \frac{9}{5}$. Moreover, if $P \in \overline{E} \backslash \overline{C}$ or $P \in \overline{E} \cap \overline{C}$ then

$$h(v) = \begin{cases} \frac{v^2}{32} \text{ if } v \in [0,1], \\ \frac{(4-v)^2}{288} \text{ if } v \in [1,4]. \end{cases} \quad \text{or } h(v) = \begin{cases} \frac{v^2}{32} \text{ if } v \in [0,1], \\ \frac{(4-v)(7v-4)}{288} \text{ if } v \in [1,4]. \end{cases}$$

So $S(W^{\overline{E}}_{\bullet,\bullet}; O) = \frac{1}{12}$ or $S(W^{\overline{E}}_{\bullet,\bullet}; O) = \frac{1}{3}$. On the other hand:

$$\delta_P(S) \geqslant \min\left\{ \frac{9}{5}, \inf_{O \in \overline{E}} \frac{A_{\overline{E},\Delta_{\overline{E}}}(O)}{S(W^{\overline{E}}_{\bullet,\bullet}; O)} \right\},$$

where $\Delta_{\overline{E}} = \frac{1}{2}P_1 + \frac{2}{3}P_2$. So we have

$$\frac{A_{\overline{E},\Delta_{\overline{E}}}(O)}{S(W^{\overline{E}}_{\bullet,\bullet};O)} = \begin{cases} 3 \text{ if } O = \overline{E} \cap \overline{C}, \\ 3 \text{ if } O = P_1, \\ 4 \text{ if } O = P_2, \\ 12 \text{ otherwise.} \end{cases}$$

Thus, $\delta_{\mathscr{P}}(X) = \frac{9}{5}$. □

$\mathbb{A}_2$ singularity on Du Val Del Pezzo surfaces of degree 1 such that $\mathscr{C}$ is nodal

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_2$ singularity at point $\mathscr{P}$. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$ and it has a node in $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X) = \frac{12}{7}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathscr{C}$ on S and E_1 and E_2 are the exceptional divisors. We have $-K_S \sim C + E_1 + E_2$. Let P be a point on S.

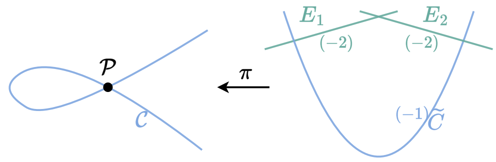


Figure 9.3: Picture: $(-K_S)^2 = 1$, $\mathbb{A}_2$ singularity (nodal)

Step 1. Suppose $P \in E_1 \cup E_2$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. Then $\tau(E_1) = 1$ and the Zariski decomposition of the divisor $-K_S - vE_1 \sim C + (1-v)E_1 + E_2$ is given by:

$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{2}E_2 \text{ if } v \in \left[0, \frac{2}{3}\right], \\ -K_S - vE_1 - (2v-1)E_2 - (3v-2)C \text{ if } v \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}E_2 \text{ if } v \in \left[0, \frac{2}{3}\right], \\ (2v-1)E_2 + (3v-2)C \text{ if } v \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{3v^2}{2} \text{ if } v \in \left[0, \frac{2}{3}\right], \\ 3(v-1)^2 \text{ if } v \in \left[\frac{2}{3}, 1\right]. \end{cases} \quad P(v) \cdot E = \begin{cases} \frac{3v}{2} \text{ if } v \in \left[0, \frac{2}{3}\right], \\ 3(1-v) \text{ if } v \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

We have $S_S(E_1) = \frac{5}{9}$. Thus, $\delta_P(S) \leq \frac{9}{5}$ for $P \in E_1 \backslash E_2$. Moreover, for such points we have

$$h(v) = \begin{cases} \frac{9v^2}{8} \text{ if } v \in \left[0, \frac{2}{3}\right], \\ \frac{3(1-v)(v+1)}{2} \text{ if } v \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet}; P) \leq \frac{14}{27} < \frac{5}{9}$. We get $\delta_P(S) = \frac{9}{5}$ for $P \in (E_1 \cup E_2) \backslash (E_1 \cap E_2)$.

Step 2. Suppose $P = E_1 \cap E_2$. Consider the blowup $\sigma : \widetilde{S} \to S$ of S at P with the exceptional divisor E_P. Suppose $\widetilde{E}_1$, $\widetilde{E}_2$ and $\widetilde{C}$ are strict transforma of E_1, E_2 and C on S. Then $\tau(E_P) = 2$ and the Zariski decomposition of the divisor $\sigma^*(-K_S) - vE_P \sim \widetilde{C} + \widetilde{E}_1 + \widetilde{E}_2 + (2-v)E_P$ is given by:

$$P(v) = \begin{cases} \sigma^*(-K_S) - vE_P - \frac{v}{3}(\widetilde{E}_1 + \widetilde{E}_2) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \sigma^*(-K_S) - vE_P - (v-1)(\widetilde{E}_1 + \widetilde{E}_2) - (2v-3)\widetilde{C} \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(\widetilde{E}_1 + \widetilde{E}_2) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ (v-1)(\widetilde{E}_1 + \widetilde{E}_2) + (2v-3)\widetilde{C} \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{3} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ (2-v)^2 \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases} \qquad P(v) \cdot E_P = \begin{cases} \frac{v}{3} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ 2 - v \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

We have $S_S(E_P) = \frac{7}{6}$. Thus, $\delta_P(S) \leq \frac{2}{7/6} = \frac{12}{7}$ for $P = E_1 \cap E_2$. Moreover,

$$h(v) \leq \begin{cases} \frac{v^2}{6} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{(2-v)v}{2} \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

Thus, $S(W^{E_P}_{\bullet,\bullet}; O) \leq \frac{7}{12}$. We get $\delta_P(S) = \frac{12}{7}$ for $P = E_1 \cap E_2$. Thus, $\delta_{\mathscr{P}}(X) = \frac{12}{7}$. □

$\mathbb{A}_2$ singularity on Du Val Del Pezzo surfaces of degree 1 such that $\mathscr{C}$ is cuspidal

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_2$ singularity at point $\mathscr{P}$. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$ and it has a cusp in $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X) = \frac{3}{2}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. We have $-K_S \sim C + E_1 + E_2$. Let P be a point on S. Let also $\sigma : \widetilde{S} \to S$ be the blowup of a point $P = E_1 \cap E_2 \cap C$. Let $\widetilde{C}$, $\widetilde{E}_1$ and $\widetilde{E}_2$ be strict transforms of C, E_1 and E_2 on $\widetilde{S}$.

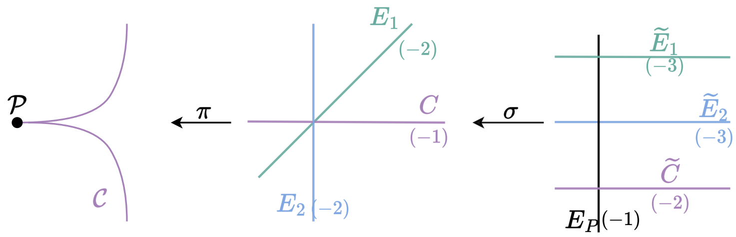


Figure 9.4: Picture: $(-K_S)^2 = 1$, $\mathbb{A}_2$ singularity (cuspidal)

Step 1. Suppose $P \in E_1 \cup E_2$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. Then $\tau(E_1) = 1$ and the Zariski decomposition of the divisor $-K_S - vE_1 \sim C + (1-v)E_1 + E_2$ is given by:

$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{2}E_2 \text{ if } v \in \left[0, \frac{2}{3}\right], \\ -K_S - vE_1 - (2v-1)E_2 - (3v-2)C \text{ if } v \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}E_2 \text{ if } v \in \left[0, \frac{2}{3}\right], \\ (2v-1)E_2 + (3v-2)C \text{ if } v \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{3v^2}{2} \text{ if } v \in \left[0, \frac{2}{3}\right], \\ 3(v-1)^2 \text{ if } v \in \left[\frac{2}{3}, 1\right]. \end{cases} \qquad P(v) \cdot E = \begin{cases} \frac{3v}{2} \text{ if } v \in \left[0, \frac{2}{3}\right], \\ 3(1-v) \text{ if } v \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

We have $S_S(E_1) = \frac{5}{9}$. Thus, $\delta_P(S) \leq \frac{9}{5}$ for $P \in E_1 \backslash E_2$. Moreover, for such points we have

$$h(v) = \begin{cases} \frac{9v^2}{8} \text{ if } v \in \left[0, \frac{2}{3}\right], \\ \frac{3(1-v)(v+1)}{2} \text{ if } v \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet}; P) \leq \frac{14}{27} < \frac{5}{9}$. We get $\delta_P(S) = \frac{9}{5}$ for $P \in (E_1 \cup E_2) \backslash (E_1 \cap E_2)$.

Step 2. Suppose $P = E_1 \cap E_2$. Consider the blowup $\sigma : \widetilde{S} \to S$ of S at P with the exceptional divisor E_P. Suppose $\widetilde{E}_1$, $\widetilde{E}_2$ and $\widetilde{C}$ are strict transforma of E_1, E_2 and C on S. Then $\tau(E_P) = 3$ and the Zariski decomposition of the divisor $\sigma^*(-K_S) - vE_P \sim \widetilde{C} + \widetilde{E}_1 + \widetilde{E}_2 + (3-v)E_P$ is given by:

$$P(v) = \begin{cases} \sigma^*(-K_S) - vE_P - \frac{v}{3}(\widetilde{E}_1 + \widetilde{E}_2) \text{ if } v \in [0,1], \\ \sigma^*(-K_S) - vE_P - (v-1)(\widetilde{E}_1 + \widetilde{E}_2) - \frac{v-1}{2}\widetilde{C} \text{ if } v \in [1,3]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(\widetilde{E}_1 + \widetilde{E}_2) \text{ if } v \in [0,1], \\ (v-1)(\widetilde{E}_1 + \widetilde{E}_2) + \frac{v-1}{2}\widetilde{C} \text{ if } v \in [1,3]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{3} \text{ if } v \in [0,1], \\ \frac{(3-v)^2}{6} \text{ if } v \in [1,3]. \end{cases} \quad P(v) \cdot E_P = \begin{cases} \frac{v}{3} \text{ if } v \in [0,1], \\ \frac{3-v}{6} \text{ if } v \in [1,3]. \end{cases}$$

We have $S_S(E_P) = \frac{4}{3}$. Thus, $\delta_P(S) \leq \frac{2}{4/3} = \frac{3}{2}$ for $P = E_1 \cap E_2 \cap C$. Moreover, if $O \in E_P \backslash (\widetilde{E}_1 \cup \widetilde{E}_2)$ if $O \in E_P \backslash \widetilde{C}$ we have:

$$h(v) \leq \begin{cases} \frac{v^2}{18} \text{ if } v \in [0,1], \\ \frac{(3-v)(5v-3)}{72} \text{ if } v \in [1,3]. \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{v^2}{6} \text{ if } v \in [0,1], \\ \frac{(3-v)(v+1)}{24} \text{ if } v \in [1,3]. \end{cases}$$

Thus, $S(W^{E_P}_{\bullet,\bullet}; O) \leq \frac{1}{3} < \frac{2}{3}$ or $S(W^{E_P}_{\bullet,\bullet}; O) \leq \frac{5}{9} < \frac{2}{3}$. We get $\delta_P(S) = \frac{3}{2}$ for $P = E_1 \cap E_2$.
Thus, $\delta_{\mathscr{P}}(X) = \frac{3}{2}$. □

$\mathbb{A}_3$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_3$ singularity at point $\mathscr{P}$. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X) = \frac{3}{2}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathscr{C}$ on S and E_1, E_2 and E_3 are the exceptional divisors with the following intersection:

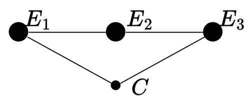


Figure 9.5: Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_3$ singularity

We have $-K_S \sim C + E_1 + E_2 + E_3$. Let P be a point on S.
Step 1. Suppose $P \in E_2$. Then $\tau(E_2) = 1$ and the Zariski decomposition of the divisor $-K_S - vE_2 \sim C + E_1 + (1-v)E_2 + E_3$ is given by:

$$P(v) = -K_S - vE_2 - \frac{v}{2}(E_1 + E_3) \text{ and } N(v) = \frac{v}{2}(E_1 + E_3) \text{ if } v \in [0,1].$$

Moreover,

$$(P(v))^2 = (1-v)(1+v) \text{ and } P(v) \cdot E_2 = v \text{ if } v \in [0,1].$$

We have $S_S(E_2) = \frac{2}{3}$. Thus, $\delta_P(S) \leq \frac{3}{2}$ for $P \in E_2$. Moreover, for such points we have $h(v) \leq v^2$ if $v \in [0,1]$. Thus, $S(W^{E_2}_{\bullet,\bullet}; P) \leq \frac{2}{3}$. We get $\delta_P(S) = \frac{3}{2}$ for $P \in E_2$.
Step 2. Suppose $P \in E_1 \cup E_3$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. Then $\tau(E_1) = 1$ and the Zariski decomposition of the divisor

$-K_S - vE_1 \sim C + (1-v)E_1 + E_2 + E_3$ is given by:

$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{3}(2E_2 + E_3) \text{ if } v \in \left[0, \frac{3}{4}\right], \\ -K_S - vE_1 - (2v-1)E_2 - (3v-2)E_3 - (4v-3)C \text{ if } v \in \left[\frac{3}{4}, 1\right] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2E_2 + E_3) \text{ if } v \in \left[0, \frac{3}{4}\right], \\ (2v-1)E_2 + (3v-2)E_3 + (4v-3)C \text{ if } v \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{4v^2}{3} \text{ if } v \in \left[0, \frac{3}{4}\right], \\ 4(v-1)^2 \text{ if } v \in \left[\frac{3}{4}, 1\right]. \end{cases} \qquad P(v) \cdot E_1 = \begin{cases} \frac{4v}{3} \text{ if } v \in \left[0, \frac{3}{4}\right], \\ 4(1-v) \text{ if } v \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

We have $S_S(E_1) = \frac{5}{9}$. Thus, $\delta_P(S) \leq \frac{9}{5}$ for $P \in E_1 \backslash E_2$. Moreover, for such points we have

$$h(v) = \begin{cases} \frac{8v^2}{9} \text{ if } v \in \left[0, \frac{3}{4}\right], \\ 4(1-v)(2v-1) \text{ if } v \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet}; P) \leq \frac{5}{12} < \frac{7}{12}$. We get $\delta_P(S) = \frac{12}{7}$ for $P \in (E_1 \cup E_3) \backslash E_2$. Thus, $\delta_{\mathscr{P}}(X) = \frac{3}{2}$. □

$\mathbb{A}_4$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_4$ singularity at point $\mathscr{P}$. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X) = \frac{4}{3}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathscr{C}$ on S and E_1, E_2, E_3 and E_4 are the exceptional divisors with the intersection:

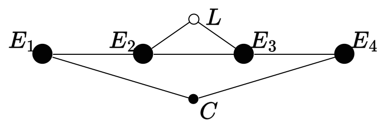


Figure 9.6: Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_4$ singularity

We have $-K_S \sim C + E_1 + E_2 + E_3 + E_4$. Let P be a point on S. Consider a linear system $\mathscr{L} = |-2K_S - (E_1 + 2E_2 + 2E_3 + E_4)|$. Using Riemann-Roch for surfaces we get $\dim|\mathscr{L}| = 1$. Thus there is a unique element $L \in |\mathscr{L}|$ such that it contains the intersection point of E_2 and E_3. Moreover we have $L \cdot E_1 = L \cdot E_4 = 0$, $L \cdot E_2 = L \cdot E_3 = 1$ and $L^2 = 0$.

Step 1. Suppose $P = E_2 \cap E_3$. Consider the blowup $\sigma : \widetilde{S} \to S$ of S at P with the exceptional divisor E_P. Suppose $\widetilde{E}_1$, $\widetilde{E}_2$, $\widetilde{E}_3$, $\widetilde{E}_4$, $\widetilde{L}$ and $\widetilde{C}$ are strict transforms of E_1, E_2, E_3, E_4, L and C on $\widetilde{S}$. Then $\tau(E_P)=\frac{5}{2}$ and the Zariski decomposition of the divisor

$$\sigma^*(-K_S)-vE_P\sim\Big(\frac{5}{2}-v\Big)E_P+\frac{1}{2}\widetilde{L}+\widetilde{E}_1+\frac{3}{2}\widetilde{E}_2+\frac{3}{2}\widetilde{E}_3+\widetilde{E}_4$$

is given by:

$$P(v)=\begin{cases}\sigma^*(-K_S)-vE_P-\frac{v}{5}(\widetilde{E}_1+2\widetilde{E}_2+2\widetilde{E}_3+\widetilde{E}_4)\text{ if }v\in[0,2],\\ \sigma^*(-K_S)-vE_P-\frac{v}{5}(\widetilde{E}_1+2\widetilde{E}_2+2\widetilde{E}_3+\widetilde{E}_4)-(v-2)\widetilde{L}\text{ if }v\in\big[2,\frac{5}{2}\big].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{5}(\widetilde{E}_1+2\widetilde{E}_2+2\widetilde{E}_3+\widetilde{E}_4)\text{ if }v\in[0,2],\\ \frac{v}{5}(\widetilde{E}_1+2\widetilde{E}_2+2\widetilde{E}_3+\widetilde{E}_4)-(v-2)\widetilde{L}\text{ if }v\in\big[2,\frac{5}{2}\big].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{v^2}{5}\text{ if }v\in[0,2],\\ \frac{(5-2v)^2}{5}\text{ if }v\in\big[2,\frac{5}{2}\big].\end{cases}\qquad P(v)\cdot E_P=\begin{cases}\frac{v}{5}\text{ if }v\in[0,2],\\ 2(1-\frac{2v}{5})\text{ if }v\in\big[2,\frac{5}{2}\big].\end{cases}$$

We have $S_S(E_P)=\frac{3}{2}$. Thus, $\delta_P(S)\leq\frac{2}{3/2}=\frac{4}{3}$ for $P=E_2\cap E_3$. Moreover, if $O\in E_P\backslash(\widetilde{E}_2\cup\widetilde{E}_3)$ if $O\in E_P\backslash\widetilde{L}$ we have:

$$h(v)\leq\begin{cases}\frac{v^2}{50}\text{ if }v\in[0,2],\\ \frac{2(5-2v)(3v-5)}{25}\text{ if }v\in\big[2,\frac{5}{2}\big].\end{cases}\qquad\text{or }h(v)\leq\begin{cases}\frac{v^2}{10}\text{ if }v\in[0,2],\\ \frac{2(5-2v)}{5}\text{ if }v\in\big[2,\frac{5}{2}\big].\end{cases}$$

Thus, $S(W^{E_P}_{\bullet,\bullet};O)\leq\frac{1}{6}<\frac{3}{4}$ or $S(W^{E_P}_{\bullet,\bullet};O)\leq\frac{11}{15}<\frac{3}{4}$. We get $\delta_P(S)=\frac{4}{3}$ for $P=E_2\cap E_3$.

Step 2. Suppose $P\in E_2\cup E_3$. Without loss of generality we can assume that $P\in E_2$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:

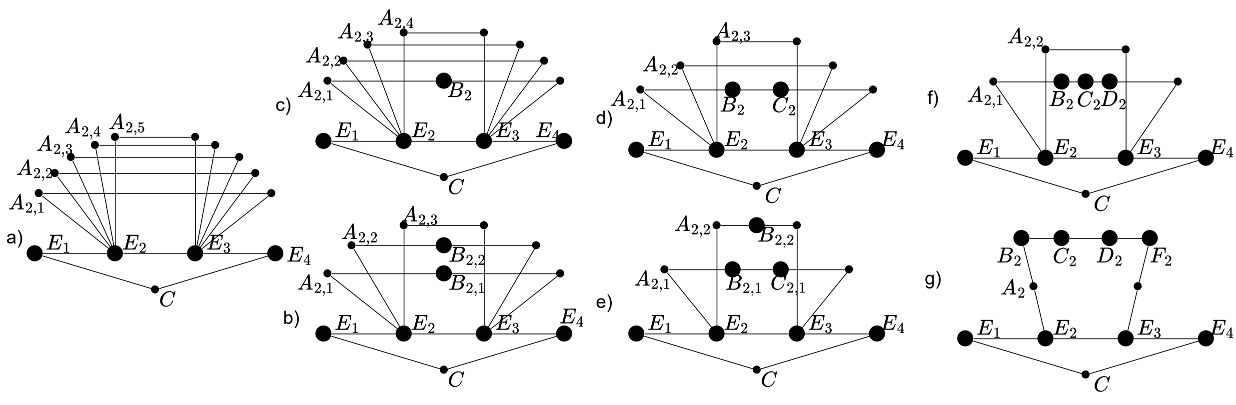


Figure 9.7: Dual graph: $(-K_S)^2=1$, $\mathbb{A}_4$ singularity, $\delta_P(S)=\frac{15}{11}$

Then $\tau(E_2)=\frac{6}{5}$ and the Zariski Decomposition of the divisor $-K_S-vE_2$ is:

a).
$$P(v)=\begin{cases}-K_S-vE_2-\frac{v}{6}(3E_1+4E_3+2E_4) \text{ if } v\in[0,1],\\ -K_S-vE_2-\frac{v}{6}(3E_1+4E_3+2E_4)-(v-1)(A_{2,1}+A_{2,2}+A_{2,3}+A_{2,4}+A_{2,5}) \text{ if } v\in\left[1,\frac{6}{5}\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{6}(3E_1+4E_3+2E_4) \text{ if } v\in[0,1],\\ \frac{v}{6}(3E_1+4E_3+2E_4)+(v-1)(A_{2,1}+A_{2,2}+A_{2,3}+A_{2,4}+A_{2,5}) \text{ if } v\in\left[1,\frac{6}{5}\right].\end{cases}$$

b).
$$P(v)=\begin{cases}-K_S-vE_2-\frac{v}{6}(3E_1+4E_3+2E_4) \text{ if } v\in[0,1],\\ -K_S-vE_2-\frac{v}{6}(3E_1+4E_3+2E_4)-(v-1)(2A_{2,1}+B_2+A_{2,2}+A_{2,3}+A_{2,4}) \text{ if } v\in\left[1,\frac{6}{5}\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{6}(3E_1+4E_3+2E_4) \text{ if } v\in[0,1],\\ \frac{v}{6}(3E_1+4E_3+2E_4)+(v-1)(2A_{2,1}+B_2+A_{2,2}+A_{2,3}+A_{2,4}) \text{ if } v\in\left[1,\frac{6}{5}\right].\end{cases}$$

c).
$$P(v)=\begin{cases}-K_S-vE_2-\frac{v}{6}(3E_1+4E_3+2E_4) \text{ if } v\in[0,1],\\ -K_S-vE_2-\frac{v}{6}(3E_1+4E_3+2E_4)-(v-1)(2A_{2,1}+B_{2,1}+2A_{2,2}+B_{2,2}+A_{2,3}) \text{ if } v\in\left[1,\frac{6}{5}\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{6}(3E_1+4E_3+2E_4) \text{ if } v\in[0,1],\\ \frac{v}{6}(3E_1+4E_3+2E_4)+(v-1)(2A_{2,1}+B_{2,1}+2A_{2,2}+B_{2,2}+A_{2,3}) \text{ if } v\in\left[1,\frac{6}{5}\right].\end{cases}$$

d).
$$P(v)=\begin{cases}-K_S-vE_2-\frac{v}{6}(3E_1+4E_3+2E_4) \text{ if } v\in[0,1],\\ -K_S-vE_2-\frac{v}{6}(3E_1+4E_3+2E_4)-(v-1)(3A_{2,1}+2B_2+C_2+2A_{2,2}+A_{2,3}) \text{ if } v\in\left[1,\frac{6}{5}\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{6}(3E_1+4E_3+2E_4) \text{ if } v\in[0,1],\\ \frac{v}{6}(3E_1+4E_3+2E_4)+(v-1)(3A_{2,1}+2B_2+C_2+A_{2,2}+A_{2,3}) \text{ if } v\in\left[1,\frac{6}{5}\right].\end{cases}$$

e).
$$P(v)=\begin{cases}-K_S-vE_2-\frac{v}{6}(3E_1+4E_3+2E_4) \text{ if } v\in[0,1],\\ -K_S-vE_2-\frac{v}{6}(3E_1+4E_3+2E_4)-(v-1)(3A_{2,1}+2B_{2,1}+C_{2,1}+2A_{2,2}+B_{2,2}) \text{ if } v\in\left[1,\frac{6}{5}\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{6}(3E_1+4E_3+2E_4) \text{ if } v\in[0,1],\\ \frac{v}{6}(3E_1+4E_3+2E_4)+(v-1)(3A_{2,1}+2B_{2,1}+C_{2,1}+2A_{2,2}+B_{2,2}) \text{ if } v\in\left[1,\frac{6}{5}\right],\end{cases}$$

f).
$$P(v)=\begin{cases}-K_S-vE_2-\frac{v}{6}(3E_1+4E_3+2E_4) \text{ if } v\in[0,1],\\ -K_S-vE_2-\frac{v}{6}(3E_1+4E_3+2E_4)-(v-1)(4A_{2,1}+3B_2+2C_2+D_2+A_{2,2}) \text{ if } v\in\left[1,\frac{6}{5}\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{6}(3E_1+4E_3+2E_4) \text{ if } v\in[0,1],\\ \frac{v}{6}(3E_1+4E_3+2E_4)+(v-1)(4A_{2,1}+3B_2+2C_2+D_2+A_{2,2}) \text{ if } v\in\left[1,\frac{6}{5}\right].\end{cases}$$

g).
$$P(v)=\begin{cases}-K_S-vE_2-\frac{v}{6}(3E_1+4E_3+2E_4) \text{ if } v\in[0,1],\\ -K_S-vE_2-\frac{v}{6}(3E_1+4E_3+2E_4)-(v-1)(5A_2+4B_2+3C_2+2D_2+F_2) \text{ if } v\in\left[1,\frac{6}{5}\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{6}(3E_1+4E_3+2E_4) \text{ if } v\in[0,1],\\ \frac{v}{6}(3E_1+4E_3+2E_4)+(v-1)(5A_2+4B_2+3C_2+2D_2+F_2) \text{ if } v\in\left[1,\frac{6}{5}\right].\end{cases}$$

The Zariski Decomposition in part a). follows from

$$-K_S - vE_2 \sim_{\mathbb{R}} \Big(\frac{6}{5} - v\Big)E_2 + \frac{1}{5}\Big(3E_1 + 4E_3 + 2E_4 + A_{2,1} + A_{2,2} + A_{2,3} + A_{2,4} + A_{2,5}\Big).$$

A similar statement holds in other parts. Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{5v^2}{6} \text{ if } v \in [0,1], \\ \frac{(6-5v)^2}{6} \text{ if } v \in \big[1, \frac{6}{5}\big]. \end{cases} \quad P(v)\cdot E_2 = \begin{cases} \frac{5v}{6} \text{ if } v \in [0,1], \\ 3(1-v) \text{ if } v \in \big[1, \frac{6}{5}\big]. \end{cases}$$

We have $S_S(E_2) = \frac{11}{15}$. Thus, $\delta_P(S) \le \frac{15}{11}$ for $P \in E_2 \backslash E_3$. Moreover, if $P \in E_2 \cap E_1$ or if $P \in E_2 \backslash E_1$ for such points we have

$$h(v) \le \begin{cases} \frac{55v^2}{72} \text{ if } v \in [0,1], \\ \frac{5(5v-6)(19v-30)}{72} \text{ if } v \in \big[1, \frac{6}{5}\big]. \end{cases} \quad \text{or } h(v) \le \begin{cases} \frac{25v^2}{72} \text{ if } v \in [0,1], \\ \frac{25(5v-6)(6-7v)}{72} \text{ if } v \in \big[1, \frac{6}{5}\big]. \end{cases}$$

Thus, $S(W^{E_2}_{\bullet,\bullet};P) \le \frac{29}{45} < \frac{11}{15}$ or $S(W^{E_2}_{\bullet,\bullet};P)\frac{1}{3} < \frac{11}{15}$. We get $\delta_P(S) = \frac{15}{11}$ for $P \in (E_1 \cup E_2) \backslash (E_1 \cap E_2)$.
Step 3. Suppose $P \in E_1 \cup E_4$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. Then $\tau(E_1) = 1$ and the Zariski decomposition of the divisor $-K_S - vE_1 \sim C + (1-v)E_1 + E_2 + E_3 + E_4$ is given by:

$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{4}(3E_2 + 2E_3 + E_4) \text{ if } v \in \big[0, \frac{4}{5}\big], \\ -K_S - vE_1 - (2v-1)E_2 - (3v-2)E_3 - (4v-3)E_4 - (5v-4)C \text{ if } v \in \big[\frac{4}{5}, 1\big]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(3E_2 + 2E_3 + E_4) \text{ if } v \in \big[0, \frac{4}{5}\big], \\ (2v-1)E_2 + (3v-2)E_3 + (4v-3)E_4 + (5v-4)C \text{ if } v \in \big[\frac{4}{5}, 1\big]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{5v^2}{4} \text{ if } v \in \big[0, \frac{4}{5}\big], \\ 5(v-1)^2 \text{ if } v \in \big[\frac{4}{5}, 1\big]. \end{cases} \quad P(v)\cdot E_1 = \begin{cases} \frac{5v}{4} \text{ if } v \in \big[0, \frac{4}{5}\big], \\ 5(1-v) \text{ if } v \in \big[\frac{4}{5}, 1\big]. \end{cases}$$

We have $S_S(E_1) = \frac{3}{5}$ Thus, $\delta_P(S) \le \frac{5}{3}$ for $P \in E_1 \backslash E_2$. Moreover, for such points we have

$$h(v) = \begin{cases} \frac{25v^2}{32} \text{ if } v \in \big[0, \frac{4}{5}\big], \\ \frac{5(1-v)(5v-3)}{2} \text{ if } v \in \big[\frac{4}{5}, 1\big]. \end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet};P) \le \frac{2}{5} < \frac{3}{5}$. We get $\delta_P(S) = \frac{5}{3}$ for $P \in (E_1 \cup E_4) \backslash (E_2 \cup E_3)$. Thus, $\delta_{\mathscr{P}}(X) = \frac{4}{3}$. □

$\mathbb{A}_5$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_5$ singularity at point $\mathscr{P}$. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X) = \frac{6}{5}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathscr{C}$ on S and E_1, E_2, E_3, E_4 and E_5 are the exceptional divisors with the intersection:

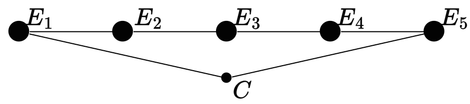


Figure 9.8: Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_5$ singularity

We have $-K_S \sim C + E_1 + E_2 + E_3 + E_4 + E_5$. Let P be a point on S.
Step 1. Suppose $P \in E_3$. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:

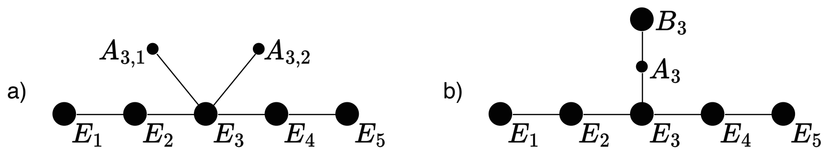


Figure 9.9: Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_5$ singularity, $\delta_P(S) = \frac{6}{5}$

Then $\tau(E_3) = \frac{3}{2}$ and the Zariski Decomposition of the divisor $-K_S - vE_3$ is:

a). $$P(v) = \begin{cases} -K_S - vE_3 - \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) \text{ if } v \in [0,1], \\ -K_S - vE_3 - \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) - (v-1)(A_{3,1} + A_{3,2}) \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) \text{ if } v \in [0,1], \\ \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) + (v-1)(A_{3,1} + A_{3,2}) \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

b). $$P(v) = \begin{cases} -K_S - vE_3 - \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) \text{ if } v \in [0,1], \\ -K_S - vE_3 - \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) - (v-1)(2A_3 + B_3) \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) \text{ if } v \in [0,1], \\ \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) + (v-1)(2A_3 + B_3) \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

The Zariski Decomposition in part a). follows from

$$-K_S - vE_3 \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)E_3 + \frac{1}{2}\Big(E_1 + 2E_2 + 2E_4 + E_5 + A_{3,1} + A_{3,2}\Big).$$

A similar statement holds in other parts. Moreover,

$$(P(v))^2 = \begin{cases} 1-\frac{2v^2}{3} \text{ if } v\in[0,1], \\ \frac{(3-2v)^2}{3} \text{ if } v\in\left[1,\frac{3}{2}\right]. \end{cases} \quad P(v)\cdot E_3 = \begin{cases} \frac{2v}{3} \text{ if } v\in[0,1], \\ 2(1-\frac{2v}{3}) \text{ if } v\in\left[1,\frac{3}{2}\right]. \end{cases}$$

We have $S_S(E_3)=\frac{5}{6}$. Thus, $\delta_P(S)\leq\frac{6}{5}$ for $P\in E_3$. Moreover, if $P\in E_3\cap(E_2\cup E_4)$ or if $P\in E_3\backslash(E_2\cup E_4)$ we have

$$h(v)\leq\begin{cases} \frac{2v^2}{3} \text{ if } v\in[0,1], \\ \frac{2(3-2v)}{3} \text{ if } v\in\left[1,\frac{3}{2}\right]. \end{cases} \quad \text{or } h(v)\leq\begin{cases} \frac{2v^2}{9} \text{ if } v\in[0,1], \\ \frac{2(3-2v)(4v-3)}{9} \text{ if } v\in\left[1,\frac{3}{2}\right]. \end{cases}$$

Thus, $S(W^{E_3}_{\bullet,\bullet};P)\leq\frac{7}{9}<\frac{5}{6}$ or $S(W^{E_3}_{\bullet,\bullet};P)\leq\frac{1}{3}<\frac{5}{6}$. We get $\delta_P(S)=\frac{6}{5}$ for $P\in E_3$.

Step 2. Suppose $P\in E_2\cup E_4$. Without loss of generality we can assume that $P\in E_2$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:

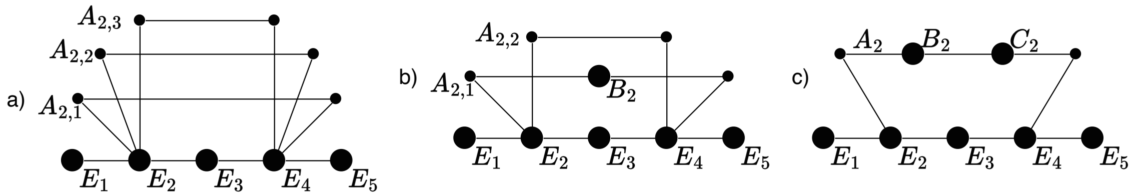


Figure 9.10: Dual graph: $(-K_S)^2=1$, $\mathbb{A}_5$ singularity, $\delta_P(S)=\frac{9}{7}$

Then $\tau(E_2)=\frac{4}{3}$ and the Zariski Decomposition of the divisor $-K_S-vE_2$ is:

a). $$P(v)=\begin{cases} -K_S-vE_2-\frac{v}{4}(2E_1+3E_3+2E_4+E_5) \text{ if } v\in[0,1], \\ -K_S-vE_2-\frac{v}{4}(2E_1+3E_3+2E_4+E_5)-(v-1)(A_{2,1}+A_{2,2}+A_{2,3}) \text{ if } v\in\left[1,\frac{4}{3}\right]. \end{cases}$$

$$N(v)=\begin{cases} \frac{v}{4}(2E_1+3E_3+2E_4+E_5) \text{ if } v\in[0,1], \\ \frac{v}{4}(2E_1+3E_3+2E_4+E_5)+(v-1)(A_{2,1}+A_{2,2}+A_{2,3}) \text{ if } v\in\left[1,\frac{4}{3}\right]. \end{cases}$$

b). $$P(v)=\begin{cases} -K_S-vE_2-\frac{v}{4}(2E_1+3E_3+2E_4+E_5) \text{ if } v\in[0,1], \\ -K_S-vE_2-\frac{v}{4}(2E_1+3E_3+2E_4+E_5)-(v-1)(2A_{2,1}+B_{2,1}+A_{2,2}) \text{ if } v\in\left[1,\frac{4}{3}\right]. \end{cases}$$

$$N(v)=\begin{cases} \frac{v}{4}(2E_1+3E_3+2E_4+E_5) \text{ if } v\in[0,1], \\ \frac{v}{4}(2E_1+3E_3+2E_4+E_5)+(v-1)(2A_{2,1}+B_{2,1}+A_{2,2}) \text{ if } v\in\left[1,\frac{4}{3}\right]. \end{cases}$$

c). $$P(v)=\begin{cases} -K_S-vE_2-\frac{v}{4}(2E_1+3E_3+2E_4+E_5) \text{ if } v\in[0,1], \\ -K_S-vE_2-\frac{v}{4}(2E_1+3E_3+2E_4+E_5)-(v-1)(3A_2+B_2+C_2) \text{ if } v\in\left[1,\frac{4}{3}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(2E_1+3E_3+2E_4+E_5) \text{ if } v\in[0,1], \\ \frac{v}{4}(2E_1+3E_3+2E_4+E_5)+(v-1)(3A_2+B_2+C_2) \text{ if } v\in\left[1,\frac{4}{3}\right]. \end{cases}$$

The Zariski Decomposition in part a). follows from

$$-K_S - vE_2 \sim_{\mathbb{R}} \Big(\frac{4}{3}-v\Big)E_2 + \frac{1}{3}\Big(2E_1+3E_3+2E_4+E_5+A_{2,1}+A_{2,2}+A_{2,3}\Big).$$

A similar statement holds in other parts. Moreover,

$$(P(v))^2 = \begin{cases} 1-\frac{3v^2}{4} \text{ if } v\in[0,1], \\ \frac{(4-3v)^2}{4} \text{ if } v\in\left[1,\frac{4}{3}\right]. \end{cases} \quad P(v)\cdot E_2 = \begin{cases} \frac{3v}{4} \text{ if } v\in[0,1], \\ 3(1-\frac{3v}{4}) \text{ if } v\in\left[1,\frac{4}{3}\right]. \end{cases}$$

We have $S_S(E_2)=\frac{7}{9}$. Thus, $\delta_P(S)\le\frac{9}{7}$ for $P\in E_2$. Moreover, if $P\in E_2\cap E_1$ or if $P\in E_2\backslash(E_1\cup E_3)$ we have

$$h(v)\le \begin{cases} \frac{21v^2}{32} \text{ if } v\in[0,1], \\ \frac{3(3v-4)(5v-12)}{32} \text{ if } v\in\left[1,\frac{4}{3}\right]. \end{cases} \quad \text{or } h(v)\le \begin{cases} \frac{9v^2}{32} \text{ if } v\in[0,1], \\ \frac{9(3v-4)(4-5v)}{32} \text{ if } v\in\left[1,\frac{4}{3}\right]. \end{cases}$$

Thus, $S(W^{E_2}_{\bullet,\bullet};P)\le\frac{23}{36}<\frac{7}{9}$ or $S(W^{E_2}_{\bullet,\bullet};P)\le\frac{1}{3}<\frac{7}{9}$. We get $\delta_P(S)=\frac{9}{7}$ for $P\in(E_2\cup E_4)\backslash E_3$.

Step 3. Suppose $P\in E_1\cup E_5$. Without loss of generality we can assume that $P\in E_1$ since the proof is similar in other cases. Then $\tau(E_1)=1$ and the Zariski decomposition of the divisor $-K_S-vE_1\sim C+(1-v)E_1+E_2+E_3+E_4+E_5$ is given by:

$$P(v) = \begin{cases} -K_S-vE_1-\frac{v}{5}(4E_2+3E_3+2E_4+E_5) \text{ if } v\in\left[0,\frac{5}{6}\right], \\ -K_S-vE_1-(2v-1)E_2-(3v-2)E_3-(4v-3)E_4-(5v-4)E_5-(6v-5)C \text{ if } v\in\left[\frac{5}{6},1\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(4E_2+3E_3+2E_4+E_5) \text{ if } v\in\left[0,\frac{5}{6}\right], \\ (2v-1)E_2+(3v-2)E_3+(4v-3)E_4+(5v-4)E_5+(6v-5)C \text{ if } v\in\left[\frac{5}{6},1\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1-\frac{6v^2}{5} \text{ if } v\in\left[0,\frac{5}{6}\right], \\ 6(v-1)^2 \text{ if } v\in\left[\frac{5}{6},1\right]. \end{cases} \quad P(v)\cdot E_1 = \begin{cases} \frac{6v}{5} \text{ if } v\in\left[0,\frac{5}{6}\right], \\ 6(1-v) \text{ if } v\in\left[\frac{5}{6},1\right]. \end{cases}$$

We have $S_S(E_1)=\frac{11}{18}$. Thus, $\delta_P(S)\le\frac{18}{11}$ for $P\in E_1\backslash E_2$. Moreover, for such points we have

$$h(v) = \begin{cases} \frac{18v^2}{25} \text{ if } v\in\left[0,\frac{5}{6}\right], \\ 6(1-v)(3v-2) \text{ if } v\in\left[\frac{5}{6},1\right]. \end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet};P) \leq \frac{7}{18} < \frac{11}{18}$. We get $\delta_P(S) = \frac{18}{11}$ for $P \in (E_1 \cup E_5) \setminus (E_2 \cup E_4)$.
Thus, $\delta_{\mathscr{P}}(X) = \frac{6}{5}$. □

$\mathbb{A}_6$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_6$ singularity at point $\mathscr{P}$. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X) = \frac{9}{8}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathscr{C}$ on S and E_1, E_2, E_3, E_4, E_6 and E_7 are the exceptional divisors with the intersection:

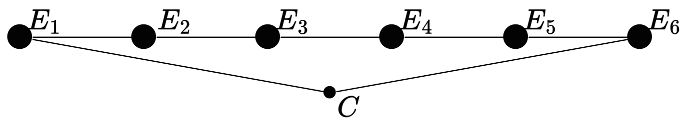


Figure 9.11: Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_6$ singularity

We have $-K_S \sim C + E_1 + E_2 + E_3 + E_4 + E_5 + E_6$. Let P be a point on S.
Step 1. Suppose $P \in E_3 \cup E_4$. Without loss of generality we can assume that $P \in E_3$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form the following dual graph:

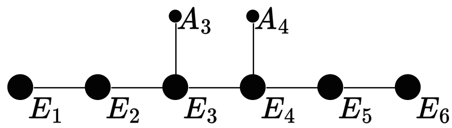


Figure 9.12: Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_6$ singularity, $\delta_P(S) = \frac{9}{8}$

Then $\tau(E_3) = \frac{3}{2}$ and the Zariski Decomposition of the divisor $-K_S - vE_3$ is:

$$P(v) = \begin{cases} -K_S - vE_3 - \frac{v}{12}(4E_1 + 8E_2 + 9E_4 + 6E_5 + 3E_6) \text{ if } v \in [0,1], \\ -K_S - vE_3 - \frac{v}{12}(4E_1 + 8E_2 + 9E_4 + 6E_5 + 3E_6) - (v-1)A_3 \text{ if } v \in \left[1, \frac{4}{3}\right], \\ -K_S - vE_3 - \frac{v}{3}(E_1 + 2E_2) - (v-1)(3E_4 + 2E_5 + E_6 + A_3) - (3v-4)A_4 \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{12}(4E_1 + 8E_2 + 9E_4 + 6E_5 + 3E_6) \text{ if } v \in [0,1], \\ \frac{v}{12}(4E_1 + 8E_2 + 9E_4 + 6E_5 + 3E_6) + (v-1)A_3 \text{ if } v \in \left[1, \frac{4}{3}\right], \\ \frac{v}{3}(E_1 + 2E_2) + (v-1)(3E_4 + 2E_5 + E_6 + A_3) + (3v-4)A_4 \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right]. \end{cases}$$

The Zariski Decomposition in part a). follows from

$$-K_S - vE_3 \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)E_3 + \frac{1}{2}\Big(E_1 + 2E_2 + 3E_4 + 2E_5 + E_6 + A_3 + A_4\Big).$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{7v^2}{12} \text{ if } v\in[0,1],\\ 2-2v+\frac{5v^2}{12} \text{ if } v\in\left[1,\frac{4}{3}\right],\\ \frac{2(3-2v)^2}{3} \text{ if } v\in\left[\frac{4}{3},\frac{3}{2}\right].\end{cases}\quad P(v)\cdot E_3=\begin{cases}\frac{7v}{12} \text{ if } v\in[0,1],\\ 1-\frac{5v}{12} \text{ if } v\in\left[1,\frac{4}{3}\right],\\ 4(1-\frac{2v}{3}) \text{ if } v\in\left[\frac{4}{3},\frac{3}{2}\right].\end{cases}$$

We have $S_S(E_3)=\frac{8}{9}$. Thus, $\delta_P(S)\leq\frac{9}{8}$ for $P\in E_3$. Moreover, if $P\in E_3\cap A_3$ or if $P\in E_3\cap E_2$ or if $P\in E_3\backslash(E_2\cup A_3)$ we have

$$h(v)\leq\begin{cases}\frac{49v^2}{288} \text{ if } v\in[0,1],\\ \frac{(12-5v)(19v-12)}{288} \text{ if } v\in\left[1,\frac{4}{3}\right],\\ \frac{4(2v-3)(v-3)}{9} \text{ if } v\in\left[\frac{4}{3},\frac{3}{2}\right].\end{cases}\quad \text{or } h(v)\leq\begin{cases}\frac{161v^2}{288} \text{ if } v\in[0,1],\\ \frac{(12-5v)(11v+12)}{288} \text{ if } v\in\left[1,\frac{4}{3}\right],\\ \frac{8(2v-3)(v-3)}{9} \text{ if } v\in\left[\frac{4}{3},\frac{3}{2}\right].\end{cases}$$

$$\text{or } h(v)\leq\begin{cases}\frac{175v^2}{288} \text{ if } v\in[0,1],\\ \frac{(12-5v)(13v+12)}{288} \text{ if } v\in\left[1,\frac{4}{3}\right],\\ \frac{4(2v-3)(5v-3)}{9} \text{ if } v\in\left[\frac{4}{3},\frac{3}{2}\right].\end{cases}$$

Thus, $S(W^{E_3}_{\bullet,\bullet};P)\leq\frac{8}{27}<\frac{8}{9}$ or $S(W^{E_3}_{\bullet,\bullet};P)\leq\frac{29}{36}<\frac{8}{9}$ or $S(W^{E_3}_{\bullet,\bullet};P)\leq\frac{8}{9}$. We get $\delta_P(S)=\frac{9}{8}$ for $P\in E_3\cup E_4$.

Step 2. Suppose $P\in E_2\cup E_5$. Without loss of generality we can assume that $P\in E_2$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:

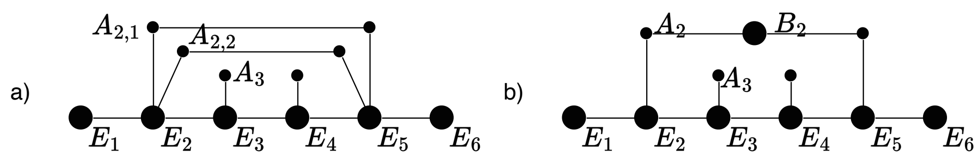


Figure 9.13: Dual graph: $(-K_S)^2=1$, $\mathbb{A}_6$ singularity, $\delta_P(S)=\frac{36}{29}$

Then $\tau(E_2)=\frac{4}{3}$ and the Zariski Decomposition of the divisor $-K_S-vE_2$ is:

a). $$P(v)=\begin{cases}-K_S-vE_2-\frac{v}{10}(5E_1+8E_3+6E_4+4E_5+2E_6) \text{ if } v\in[0,1],\\ -K_S-vE_2-\frac{v}{10}(5E_1+8E_3+6E_4+4E_5+2E_6)-(v-1)(A_{2,1}+A_{2,2}) \text{ if } v\in\left[1,\frac{5}{4}\right],\\ -K_S-vE_2-\frac{v}{2}E_1-(v-1)(4E_3+3E_4+2E_5+E_6+A_{2,1}+A_{2,2})-(4v-5)A_3 \text{ if } v\in\left[\frac{5}{4},\frac{4}{3}\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{10}(5E_1+8E_3+6E_4+4E_5+2E_6) \text{ if } v\in[0,1],\\ \frac{v}{10}(5E_1+8E_3+6E_4+4E_5+2E_6)+(v-1)(A_{2,1}+A_{2,2}) \text{ if } v\in\left[1,\frac{5}{4}\right],\\ \frac{v}{2}E_1+(v-1)(4E_3+3E_4+2E_5+E_6+A_{2,1}+A_{2,2})+(4v-5)A_3 \text{ if } v\in\left[\frac{5}{4},\frac{4}{3}\right].\end{cases}$$

b). $$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{10}(5E_1 + 8E_3 + 6E_4 + 4E_5 + 2E_6) \text{ if } v \in [0,1], \\ -K_S - vE_2 - \frac{v}{10}(5E_1 + 8E_3 + 6E_4 + 4E_5 + 2E_6) - (v-1)(2A_2 + B_2) \text{ if } v \in \left[1, \frac{5}{4}\right], \\ -K_S - vE_2 - \frac{v}{2}E_1 - (v-1)(4E_3 + 3E_4 + 2E_5 + E_6 + 2A_2 + B_2) - (4v-5)A_3 \text{ if } v \in \left[\frac{5}{4}, \frac{4}{3}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{10}(5E_1 + 8E_3 + 6E_4 + 4E_5 + 2E_6) \text{ if } v \in [0,1], \\ \frac{v}{10}(5E_1 + 8E_3 + 6E_4 + 4E_5 + 2E_6) + (v-1)(2A_2 + B_2) \text{ if } v \in \left[1, \frac{5}{4}\right], \\ \frac{v}{2}E_1 + (v-1)(4E_3 + 3E_4 + 2E_5 + E_6 + 2A_2 + B_2) + (4v-5)A_3 \text{ if } v \in \left[\frac{5}{4}, \frac{4}{3}\right]. \end{cases}$$

The Zariski Decomposition in part a). follows from

$$-K_S - vE_2 \sim_{\mathbb{R}} \Big(\frac{4}{3} - v\Big)E_2 + \frac{1}{3}\Big(2E_1 + 4E_3 + 3E_4 + 2E_5 + E_6 + A_{2,1} + A_{2,2} + A_3\Big).$$

A similar statement holds in other parts. Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{7v^2}{10} \text{ if } v \in [0,1], \\ 3 - 4v + \frac{13v^2}{10} \text{ if } v \in \left[1, \frac{5}{4}\right], \\ \frac{(4-3v)^2}{2} \text{ if } v \in \left[\frac{5}{4}, \frac{4}{3}\right]. \end{cases} \quad P(v)\cdot E_2 = \begin{cases} \frac{7v}{10} \text{ if } v \in [0,1], \\ 2 - \frac{13v}{10} \text{ if } v \in \left[1, \frac{5}{4}\right], \\ 3(2 - \frac{3v}{2}) \text{ if } v \in \left[\frac{5}{4}, \frac{4}{3}\right]. \end{cases}$$

We have $S_S(E_2) = \frac{29}{36}$. Thus, $\delta_P(S) \le \frac{36}{29}$ for $P \in E_2$. Moreover, if $P \in E_2 \cap E_1$ or if $P \in E_2 \backslash (E_1 \cup E_3)$ we have

$$h(v) \le \begin{cases} \frac{119v^2}{200} \text{ if } v \in [0,1], \\ \frac{(13v-20)(3v-20)}{200} \text{ if } v \in \left[1, \frac{5}{4}\right], \\ \frac{3(3v-4)(7v-12)}{8} \text{ if } v \in \left[\frac{5}{4}, \frac{4}{3}\right]. \end{cases} \quad \text{or } h(v) \le \begin{cases} \frac{49v^2}{200} \text{ if } v \in [0,1], \\ \frac{(13v-20)(27v-20)}{200} \text{ if } v \in \left[1, \frac{5}{4}\right], \\ \frac{3(3v-4)(v-4)}{8} \text{ if } v \in \left[\frac{5}{4}, \frac{4}{3}\right]. \end{cases}$$

Thus $S(W^{E_2}_{\bullet,\bullet}; P) \le \frac{29}{45} < \frac{29}{36}$ or $S(W^{E_2}_{\bullet,\bullet}; P) \le \frac{23}{72} < \frac{29}{36}$. We get $\delta_P(S) = \frac{36}{29}$ for $P \in (E_2 \cup E_5) \backslash (E_3 \cup E_4)$.

Step 3. Suppose $P \in E_1 \cup E_6$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. Then $\tau(E_1) = 1$ and the Zariski decomposition of the divisor $-K_S - vE_1 \sim C + (1-v)E_1 + E_2 + E_3 + E_4 + E_5 + E_6$ is given by:

$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{6}(5E_2 + 4E_3 + 3E_4 + 2E_5 + E_6) \text{ if } v \in \left[0, \frac{6}{7}\right], \\ -K_S - vE_1 - (2v-1)E_2 - (3v-2)E_3 - (4v-3)E_4 - (5v-4)E_5 - (6v-5)E_6 - (7v-6)C \text{ if } v \in \left[\frac{6}{7}, 1\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(5E_2 + 4E_3 + 3E_4 + 2E_5 + E_6) \text{ if } v \in \left[0, \frac{6}{7}\right], \\ (2v-1)E_2 + (3v-2)E_3 + (4v-3)E_4 + (5v-4)E_5 + (6v-5)E_6 + (7v-6)C \text{ if } v \in \left[\frac{6}{7}, 1\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1-\frac{7v^2}{6} \text{ if } v \in \left[0,\frac{6}{7}\right], \\ 7(v-1)^2 \text{ if } v \in \left[\frac{6}{7},1\right]. \end{cases} \quad P(v)\cdot E_1 = \begin{cases} \frac{7v}{6} \text{ if } v \in \left[0,\frac{6}{7}\right], \\ 7(1-v) \text{ if } v \in \left[\frac{6}{7},1\right]. \end{cases}$$

We have $S_S(E_1) = \frac{13}{21}$. Thus, $\delta_P(S) \le \frac{21}{13}$ for $P \in E_1 \backslash E_2$. Moreover, for such points we have

$$h(v) = \begin{cases} \frac{49v^2}{72} \text{ if } v \in \left[0,\frac{6}{7}\right], \\ \frac{7(1-v)(7v-5)}{2} \text{ if } v \in \left[\frac{6}{7},1\right]. \end{cases}$$

Thus $S(W^{E_1}_{\bullet,\bullet};P) \le \frac{8}{21} < \frac{13}{21}$. We get $\delta_P(S) = \frac{21}{13}$ for $P \in (E_1 \cup E_6)\backslash(E_2 \cup E_5)$.
Thus, $\delta_{\mathscr{P}}(X) = \frac{9}{8}$. □

$\mathbb{A}_7$ singularity & reducible ramification divisor on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_7$ singularity at point $\mathscr{P}$. X can be realized as the double cover $X \xrightarrow{2:1} \mathbb{P}(1,1,2)$, which is ramified along a sextic curve $R \in \mathbb{P}(1,1,2)$. Suppose R is reducible. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X) = 1$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathscr{C}$ on S and E_1, E_2, E_3, E_4, E_5, E_6 and E_7 are the exceptional divisors with the intersection:

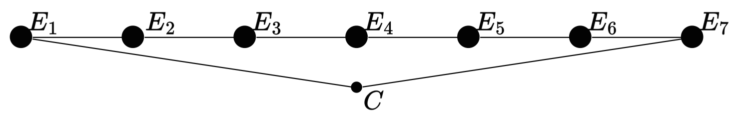


Figure 9.14: Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_7$ singularity (reducible ramification divisor)

We have $-K_S \sim C+E_1+E_2+E_3+E_4+E_5+E_6+E_7$. Let P be a point on S. If the ramification divisor R is reducible, then this implies the existence of a (-1)-curve A_4 which intersects the fundamental cycle only at E_4 and this intersection is transversal.

Step 1. Suppose $P \in E_4$. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form the following dual graph:

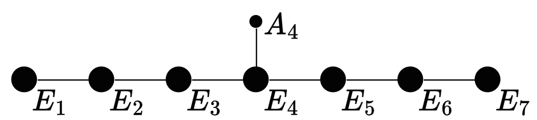


Figure 9.15: Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_7$ singularity, $\delta_P(S) = 1$

Then $\tau(E_4)=2$ and the Zariski Decomposition of the divisor $-K_S-vE_4$ is:

$$P(v)=\begin{cases}-K_S-vE_4-\frac{v}{4}(E_1+2E_2+3E_3+3E_5+2E_6+E_7) \text{ if } v\in[0,1],\\ -K_S-vE_4-\frac{v}{4}(E_1+2E_2+3E_3+3E_5+2E_6+E_7)-(v-1)A_4 \text{ if } v\in[1,2].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{4}(E_1+2E_2+3E_3+3E_5+2E_6+E_7) \text{ if } v\in[0,1],\\ \frac{v}{4}(E_1+2E_2+3E_3+3E_5+2E_6+E_7)+(v-1)A_4 \text{ if } v\in[1,2].\end{cases}$$

The Zariski Decomposition follows from

$$-K_S-vE_4\sim_{\mathbb{R}}(2-v)E_4+\frac{1}{4}\Big(E_1+2E_2+3E_3+3E_5+2E_6+E_7+4A_4\Big).$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{v^2}{2} \text{ if } v\in[0,1],\\ \frac{(2-v)^2}{2} \text{ if } v\in[1,2].\end{cases}\quad P(v)\cdot E_4=\begin{cases}\frac{v}{2} \text{ if } v\in[0,1],\\ 1-\frac{v}{2}) \text{ if } v\in[1,2].\end{cases}$$

We have $S_S(E_4)=1$. Thus, $\delta_P(S)\le 1$ for $P\in E_4$. Moreover, if $P\in E_4\cap(E_3\cup E_5)$ or if $P\in E_4\backslash(E_3\cup E_5)$ we have

$$h(v)\le\begin{cases}\frac{v^2}{2} \text{ if } v\in[0,1],\\ \frac{(2-v)(v+1)}{4} \text{ if } v\in[1,2].\end{cases}\quad \text{or } h(v)\le\begin{cases}\frac{v^2}{8} \text{ if } v\in[0,1],\\ \frac{(2-v)(3v-2)}{8} \text{ if } v\in[1,2].\end{cases}$$

Thus $S(W^{E_4}_{\bullet,\bullet};P)\le\frac{11}{12}<1$ or $S(W^{E_4}_{\bullet,\bullet};P)\le\frac{1}{3}<1$. We get $\delta_P(S)=1$ for $P\in E_4$.

Step 2. Suppose $P\in E_3\cup E_5$. Without loss of generality we can assume that $P\in E_3$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form the following dual graph:

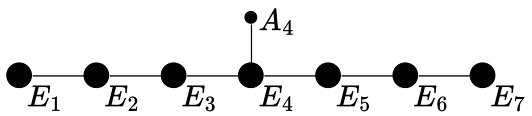


Figure 9.16: Dual graph: $(-K_S)^2=1$, $\mathbb{A}_7$ singularity, $\delta_P(S)=\frac{12}{11}$

Then $\tau(E_3)=\frac{3}{2}$ and the Zariski Decomposition of the divisor $-K_S-vE_3$ is:

$$P(v)=\begin{cases}-K_S-vE_3-\frac{v}{15}(5E_1+10E_2+12E_4+9E_5+6E_6+3E_7) \text{ if } v\in\big[0,\frac{5}{4}\big],\\ -K_S-vE_3-\frac{v}{3}(E_1+2E_2)-(v-1)(4E_4+3E_5+2E_6+E_7)-(4v-5)A_4 \text{ if } v\in\big[\frac{5}{4},\frac{3}{2}\big].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{15}(5E_1+10E_2+12E_4+9E_5+6E_6+3E_7) \text{ if } v\in\big[0,\frac{5}{4}\big],\\ \frac{v}{3}(E_1+2E_2)+(v-1)(4E_4+3E_5+2E_6+E_7)+(4v-5)A_4 \text{ if } v\in\big[\frac{5}{4},\frac{3}{2}\big].\end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_3 \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)E_3 + \frac{1}{2}\Big(E_1 + 2E_2 + 4E_4 + 3E_5 + 2E_6 + E_7 + 2A_4\Big).$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{8v^2}{15} \text{ if } v \in \big[0, \frac{5}{4}\big], \\ \frac{2(3-2v)^2}{3} \text{ if } v \in \big[\frac{5}{4}, \frac{3}{2}\big]. \end{cases} \quad P(v) \cdot E_3 = \begin{cases} \frac{8v}{15} \text{ if } v \in \big[0, \frac{5}{4}\big], \\ 4(1 - \frac{2v}{3}) \text{ if } v \in \big[\frac{5}{4}, \frac{3}{2}\big]. \end{cases}$$

We have $S_S(E_3) = \frac{11}{12}$. Thus, $\delta_P(S) \le \frac{12}{11}$ for $P \in E_3$. Moreover, if $P \in E_3 \backslash E_4$ we have

$$h(v) \le \begin{cases} \frac{112v^2}{225} \text{ if } v \in \big[0, \frac{5}{4}\big], \\ \frac{8(2v-3)(v-3)}{9} \text{ if } v \in \big[\frac{5}{4}, \frac{3}{2}\big]. \end{cases}$$

Thus, $S(W^{E_3}_{\bullet,\bullet}; P) \le \frac{5}{6} < \frac{11}{12}$. We get $\delta_P(S) = \frac{12}{11}$ for $P \in (E_3 \cup E_5) \backslash E_4$.

Step 3. Suppose $P \in E_2 \cup E_6$. Without loss of generality we can assume that $P \in E_2$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:

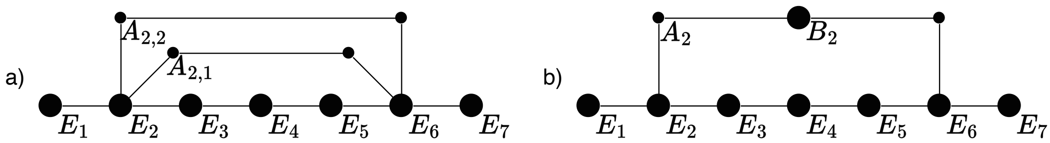


Figure 9.17: Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_7$ singularity, $\delta_P(S) = \frac{6}{5}$

Then $\tau(E_2) = \frac{3}{2}$ and the Zariski Decomposition of the divisor $-K_S - vE_2$ is:

a). $$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{6}(3E_1 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in [0, 1], \\ -K_S - vE_2 - \frac{v}{6}(3E_1 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) - (v-1)(A_{2,1} + A_{2,2}) \text{ if } v \in \big[1, \frac{3}{2}\big]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(3E_1 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in [0, 1], \\ \frac{v}{6}(3E_1 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) + (v-1)(A_{2,1} + A_{2,2}) \text{ if } v \in \big[1, \frac{3}{2}\big]. \end{cases}$$

b). $$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{6}(3E_1 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in [0, 1], \\ -K_S - vE_2 - \frac{v}{6}(3E_1 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) - (v-1)(2A_2 + B_2) \text{ if } v \in \big[1, \frac{3}{2}\big]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(3E_1 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in [0, 1], \\ \frac{v}{6}(3E_1 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) + (v-1)(2A_2 + B_2) \text{ if } v \in \big[1, \frac{3}{2}\big]. \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_2 \sim_{\mathbb{R}} \Big(\frac{3}{2}-v\Big)E_2 + \frac{1}{4}\Big(3E_1+5E_3+4E_4+3E_5+2E_6+E_7+2A_{2,1}+2A_{2,2}\Big).$$

Moreover,

$$(P(v))^2 = \begin{cases} 1-\frac{2v^2}{3} \text{ if } v\in[0,1], \\ \frac{(3-2v)^2}{3} \text{ if } v\in\big[1,\frac{3}{2}\big]. \end{cases} \quad P(v)\cdot E_2 = \begin{cases} \frac{2v}{3} \text{ if } v\in[0,1], \\ 2(1-\frac{2v}{3}) \text{ if } v\in\big[1,\frac{3}{2}\big]. \end{cases}$$

We have $S_S(E_2)=\frac{5}{6}$. Thus, $\delta_P(S)\le\frac{6}{5}$ for $P\in E_2$. Moreover, if $P\in E_2\cap E_1$ or if $P\in E_2\backslash(E_1\cup E_3)$ we have

$$h(v)\le \begin{cases} \frac{5v^2}{9} \text{ if } v\in[0,1], \\ \frac{(2v-3)(v-6)}{9} \text{ if } v\in\big[1,\frac{3}{2}\big]. \end{cases} \quad \text{or } h(v)\le \begin{cases} \frac{2v^2}{9} \text{ if } v\in[0,1], \\ \frac{2(3-2v)(4v-3)}{9} \text{ if } v\in\big[1,\frac{3}{2}\big]. \end{cases}$$

Thus $S(W^{E_2}_{\bullet,\bullet};P)\le\frac{23}{36}<\frac{5}{6}$ or $S(W^{E_2}_{\bullet,\bullet};P)\le\frac{1}{3}<\frac{5}{6}$. We get $\delta_P(S)=\frac{6}{5}$ for $P\in(E_2\cup E_6)\backslash(E_1\cup E_7)$.
Step 4. Suppose $P\in E_1\cup E_7$. Without loss of generality we can assume that $P\in E_1$ since the proof is similar in other cases. Then $\tau(E_1)=1$ and the Zariski decomposition of the divisor $-K_S-vE_1$ is given by:

$$P(v)=\begin{cases} -K_S-vE_1-\frac{v}{7}(6E_2+5E_3+4E_4+3E_5+2E_6+E_7) \text{ if } v\in\big[0,\frac{7}{8}\big], \\ -K_S-vE_1-(2v-1)E_2-(3v-2)E_3-(4v-3)E_4-(5v-4)E_5-(6v-5)E_6-(7v-6)E_7-(8v-7)C \text{ if } v\in\big[\frac{7}{8},1\big]. \end{cases}$$

$$N(v)=\begin{cases} \frac{v}{7}(6E_2+5E_3+4E_4+3E_5+2E_6+E_7) \text{ if } v\in\big[0,\frac{7}{8}\big], \\ (2v-1)E_2+(3v-2)E_3-(4v-3)E_4+(5v-4)E_5+(6v-5)E_6+(7v-6)C \text{ if } v\in\big[\frac{7}{8},1\big]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1-\frac{8v^2}{7} \text{ if } v\in\big[0,\frac{7}{8}\big], \\ 8(v-1)^2 \text{ if } v\in\big[\frac{7}{8},1\big]. \end{cases} \quad P(v)\cdot E_1 = \begin{cases} \frac{8v}{7} \text{ if } v\in\big[0,\frac{7}{8}\big], \\ 8(1-v) \text{ if } v\in\big[\frac{7}{8},1\big]. \end{cases}$$

We have $S_S(E_1)=\frac{5}{8}$. Thus, $\delta_P(S)\le\frac{8}{5}$ for $P\in E_1\backslash E_2$. Moreover, for such points we have

$$h(v)=\begin{cases} \frac{32v^2}{49} \text{ if } v\in\big[0,\frac{7}{8}\big], \\ 8(1-v)(3v-4) \text{ if } v\in\big[\frac{7}{8},1\big] \end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet};P)\le\frac{13}{96}<\frac{5}{8}$. We get $\delta_P(S)=\frac{8}{5}$ for $P\in(E_1\cup E_7)\backslash(E_2\cup E_6)$. Thus, $\delta_{\mathscr{P}}(X)=1$. $\square$

$\mathbb{A}_7$ singularity & irreducible ramification divisor on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_7$ singularity at point $\mathscr{P}$. X can be realized as the double cover $X \xrightarrow{2:1} \mathbb{P}(1,1,2)$, which is ramified along a sextic curve $R \in \mathbb{P}(1,1,2)$. Suppose R is irreducible. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X) = \frac{18}{17}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathscr{C}$ on S and E_1, E_2, E_3, E_4, E_5, E_6 and E_7 are the exceptional divisors with the intersection: We have $-K_S \sim C+E_1+E_2+E_3+E_4+E_5+E_6+E_7$.

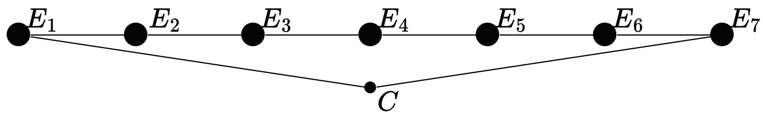


Figure 9.18: Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_7$ singularity (irreducible ramification divisor)

Let P be a point on S. If the ramification divisor R is reducible, then this implies that there is no (-1)-curve that intersects the fundamental cycle only at E_4.

Step 1. Suppose $P \in E_4$. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form the following dual graph:

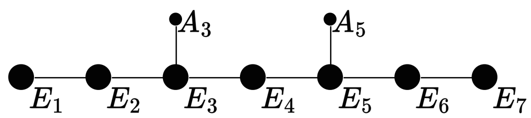


Figure 9.19: Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_7$ singularity, $\delta_P(S) = \frac{18}{17}$ (1)

Then $\tau(E_4) = \frac{3}{2}$ and the Zariski Decomposition of the divisor $-K_S - vE_4$ is:

$$P(v) = \begin{cases} -K_S - vE_4 - \frac{v}{4}(E_1+2E_2+3E_3+3E_5+2E_6+E_7) \text{ if } v \in \left[0, \frac{4}{3}\right], \\ -K_S - vE_4 - (v-1)(E_1+2E_2+3E_3+3E_5+2E_6+E_7) - (3v-4)A_3 \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(E_1+2E_2+3E_3+3E_5+2E_6+E_7) \text{ if } v \in \left[0, \frac{4}{3}\right], \\ (v-1)(E_1+2E_2+3E_3+3E_5+2E_6+E_7) + (3v-4)A_3 \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right]. \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_4 \sim_{\mathbb{R}} \left(\frac{3}{2} - v\right)E_4 + \frac{1}{2}\Big(E_1+2E_2+3E_3+3E_5+2E_6+E_7+2A_3\Big).$$

Moreover,

$$(P(v))^2 = \begin{cases} 1-\frac{v^2}{2} \text{ if } v \in \left[0, \frac{4}{3}\right], \\ (3-2v)^2 \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right]. \end{cases} \quad P(v)\cdot E_4 = \begin{cases} \frac{v}{2} \text{ if } v \in \left[0, \frac{4}{3}\right], \\ 2(3-2v) \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right]. \end{cases}$$

We have $S_S(E_4) = \frac{17}{18}$. Thus, $\delta_P(S) \le \frac{18}{17}$ for $P \in E_4$. Moreover, if $P \in E_4$ we have

$$h(v) \le \begin{cases} \frac{v^2}{2} \text{ if } v \in \left[0, \frac{4}{3}\right], \\ 2(3-2v)v \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right] \end{cases}$$

Thus $S(W^{E_2}_{\bullet,\bullet};P) \le \frac{17}{18}$. We get $\delta_P(S) = \frac{18}{17}$ for $P \in E_4$.

Step 2. Suppose $P \in E_3 \cup E_5$. Without loss of generality we can assume that $P \in E_3$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form the following dual graph:

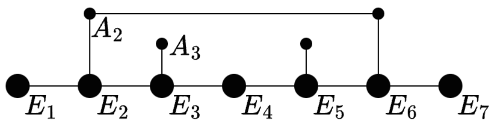


Figure 9.20: Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_7$ singularity, $\delta_P(S) = \frac{18}{17}$ (2)

Then $\tau(E_3) = \frac{5}{3}$ and the Zariski Decomposition of the divisor $-K_S - vE_3$ is:

$$P(v) = \begin{cases} -K_S - vE_3 - \frac{v}{15}(5E_1+10E_2+12E_4+9E_5+6E_6+3E_7) \text{ if } v \in [0,1], \\ -K_S - vE_3 - \frac{v}{15}(5E_1+10E_2+12E_4+9E_5+6E_6+3E_7) - (v-1)A_3 \text{ if } v \in \left[1, \frac{3}{2}\right], \\ -K_S - vE_3 - (v-1)(E_1+2E_2+A_3) - \frac{v}{5}(4E_4+3E_5+2E_6+E_7) - (2v-3)A_2 \text{ if } v \in \left[\frac{3}{2}, \frac{5}{3}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{15}(5E_1+10E_2+12E_4+9E_5+6E_6+3E_7) \text{ if } v \in [0,1], \\ \frac{v}{15}(5E_1+10E_2+12E_4+9E_5+6E_6+3E_7) + (v-1)A_3 \text{ if } v \in \left[1, \frac{3}{2}\right], \\ (v-1)(E_1+2E_2+A_3) + \frac{v}{5}(4E_4+3E_5+2E_6+E_7) + (2v-3)A_2 \text{ if } v \in \left[\frac{3}{2}, \frac{5}{3}\right]. \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_3 \sim_{\mathbb{R}} \left(\frac{5}{3} - v\right)E_3 + \frac{1}{3}\Big(2E_1+4E_2+2A_3+4E_4+3E_5+2E_6+E_7+2A_2\Big).$$

Moreover,

$$(P(v))^2 = \begin{cases} 1-\frac{8v^2}{15} \text{ if } v\in[0,1], \\ 2-2v+\frac{7v^2}{15} \text{ if } v\in\left[1,\frac{3}{2}\right], \\ \frac{(5-3v)^2}{5} \text{ if } v\in\left[\frac{3}{2},\frac{5}{3}\right]. \end{cases} \quad P(v)\cdot E_3 = \begin{cases} \frac{8v}{15} \text{ if } v\in[0,1], \\ 1-\frac{7v}{15} \text{ if } v\in\left[1,\frac{3}{2}\right], \\ 3(1-\frac{3v}{5}) \text{ if } v\in\left[\frac{3}{2},\frac{5}{3}\right]. \end{cases}$$

We have $S_S(E_3)=\frac{17}{18}$. Thus, $\delta_P(S)\leq\frac{18}{17}$ for $P\in E_3$. Moreover, if $P\in E_3\cap A_3$ or if $P\in E_3\cap E_2$ we have

$$h(v)\leq\begin{cases} \frac{32v^2}{225} \text{ if } v\in[0,1], \\ \frac{(15-7v)(23v-15)}{450} \text{ if } v\in\left[1,\frac{3}{2}\right], \\ \frac{3(5-3v)(v+5)}{50} \text{ if } v\in\left[\frac{3}{2},\frac{5}{3}\right]. \end{cases} \quad \text{or } h(v)\leq\begin{cases} \frac{112v^2}{225} \text{ if } v\in[0,1], \\ \frac{(15-7v)(13v+15)}{450} \text{ if } v\in\left[1,\frac{3}{2}\right], \\ \frac{3(5-3v)(11v-5)}{50} \text{ if } v\in\left[\frac{3}{2},\frac{5}{3}\right]. \end{cases}$$

Thus $S(W^{E_3}_{\bullet,\bullet};P)\leq\frac{14}{45}<\frac{17}{18}$ or $S(W^{E_3}_{\bullet,\bullet};P)\leq\frac{37}{45}<\frac{17}{18}$. We get $\delta_P(S)=\frac{18}{17}$ for $P\in(E_3\cup E_5)\backslash E_4$.

Step 3. Suppose $P\in E_2\cup E_6$. Without loss of generality we can assume that $P\in E_2$ since the proof is similar in other cases. Then $\tau(E_2)=\frac{4}{3}$ and the Zariski Decomposition of the divisor $-K_S-vE_2$ is:

$$P(v)=\begin{cases} -K_S-vE_2-\frac{v}{6}(3E_1+5E_3+4E_4+3E_5+2E_6+E_7) \text{ if } v\in[0,1], \\ -K_S-vE_2-\frac{v}{6}(3E_1+5E_3+4E_4+3E_5+2E_6+E_7)-(v-1)A_2 \text{ if } v\in\left[1,\frac{6}{5}\right], \\ -K_S-vE_2-\frac{v}{2}E_1-(v-1)(5E_3+4E_4+3E_5+2E_6+E_7+A_2)-(5v-6)A_3 \text{ if } v\in\left[\frac{6}{5},\frac{4}{3}\right]. \end{cases}$$

$$N(v)=\begin{cases} \frac{v}{6}(3E_1+5E_3+4E_4+3E_5+2E_6+E_7) \text{ if } v\in[0,1], \\ \frac{v}{6}(3E_1+5E_3+4E_4+3E_5+2E_6+E_7)+(v-1)A_2 \text{ if } v\in\left[1,\frac{6}{5}\right], \\ \frac{v}{2}E_1+(v-1)(5E_3+4E_4+3E_5+2E_6+E_7+A_2)+(5v-6)A_3 \text{ if } v\in\left[\frac{6}{5},\frac{4}{3}\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1-\frac{2v^2}{3} \text{ if } v\in[0,1], \\ 2-2v+\frac{v^2}{3} \text{ if } v\in\left[1,\frac{6}{5}\right], \\ \frac{(4-3v)^2}{2} \text{ if } v\in\left[\frac{6}{5},\frac{4}{3}\right]. \end{cases} \quad P(v)\cdot E_2 = \begin{cases} \frac{2v}{3} \text{ if } v\in[0,1], \\ 1-\frac{v}{3} \text{ if } v\in\left[1,\frac{6}{5}\right], \\ 3(2-\frac{3v}{2}) \text{ if } v\in\left[\frac{6}{5},\frac{4}{3}\right]. \end{cases}$$

The Zariski Decomposition follows from

$$-K_S-vE_2\sim_{\mathbb{R}}\Big(\frac{4}{3}-v\Big)E_2+\frac{1}{3}\Big(2E_1+5E_3+4E_4+3E_5+2E_6+E_7+A_2+2A_3\Big).$$

We have $S_S(E_2)=\frac{37}{45}$. Thus, $\delta_P(S)\leq\frac{45}{37}$ for $P\in E_2$. Moreover, if $P\in E_2\cap E_1$ or if $P\in E_2\backslash(E_1\cup E_3)$ we have

$$h(v)\leq\begin{cases}\frac{5v^2}{9} \text{ if } v\in[0,1],\\ \frac{(3-v)(2v+3)}{18} \text{ if } v\in\left[1,\frac{6}{5}\right],\\ \frac{3(3v-4)(7v-12)}{8} \text{ if } v\in\left[\frac{6}{5},\frac{4}{3}\right].\end{cases} \quad \text{or } h(v)\leq\begin{cases}\frac{2v^2}{9} \text{ if } v\in[0,1],\\ \frac{(3-v)(5v-3)}{18} \text{ if } v\in\left[1,\frac{6}{5}\right],\\ \frac{3(3v-4)(5v-8)}{8} \text{ if } v\in\left[\frac{6}{5},\frac{4}{3}\right].\end{cases}$$

Thus, $S(W^{E_2}_{\bullet,\bullet};P)\leq\frac{59}{90}<\frac{37}{45}$ or $S(W^{E_2}_{\bullet,\bullet};P)\leq\frac{13}{45}<\frac{37}{45}$. We get $\delta_P(S)=\frac{45}{37}$ for $P\in(E_2\cup E_6)\backslash(E_3\cup E_5)$.

Step 4. Suppose $P\in E_1\cup E_7$. Without loss of generality we can assume that $P\in E_1$ since the proof is similar in other cases. Then $\tau(E_1)=1$ and the Zariski decomposition of the divisor $-K_S-vE_1\sim C+E_1+E_2+E_3+E_4+E_5+E_6+E_7$ is given by:

$$P(v)=\begin{cases}-K_S-vE_1-\frac{v}{7}(6E_2+5E_3+4E_4+3E_5+2E_6+E_7) \text{ if } v\in\left[0,\frac{7}{8}\right],\\ -K_S-vE_1-(2v-1)E_2-(3v-2)E_3-(4v-3)E_4-(5v-4)E_5-(6v-5)E_6-(7v-6)E_7-(8v-7)C \text{ if } v\in\left[\frac{7}{8},1\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{7}(6E_2+5E_3+4E_4+3E_5+2E_6+E_7) \text{ if } v\in\left[0,\frac{7}{8}\right],\\ (2v-1)E_2+(3v-2)E_3+(4v-3)E_4+(5v-4)E_5+(6v-5)E_6+(7v-6)C \text{ if } v\in\left[\frac{7}{8},1\right].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{8v^2}{7} \text{ if } v\in\left[0,\frac{7}{8}\right],\\ 8(v-1)^2 \text{ if } v\in\left[\frac{7}{8},1\right].\end{cases} \quad P(v)\cdot E_1=\begin{cases}\frac{8v}{7} \text{ if } v\in\left[0,\frac{7}{8}\right],\\ 8(1-v) \text{ if } v\in\left[\frac{7}{8},1\right].\end{cases}$$

We have $S_S(E_1)=\frac{5}{8}$. Thus, $\delta_P(S)\leq\frac{8}{5}$ for $P\in E_1\backslash E_2$. Moreover, for such points we have

$$h(v)=\begin{cases}\frac{32v^2}{49} \text{ if } v\in\left[0,\frac{7}{8}\right],\\ 8(1-v)(3v-4) \text{ if } v\in\left[\frac{7}{8},1\right]\end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet};P)\leq\frac{13}{96}<\frac{5}{8}$. We get $\delta_P(S)=\frac{8}{5}$ for $P\in(E_1\cup E_7)\backslash(E_2\cup E_6)$. Thus, $\delta_{\mathscr{P}}(X)=\frac{18}{17}$. □

$\mathbb{A}_8$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_8$ singularity at point $\mathscr{P}$. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X)=1$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathscr{C}$ on S and E_1, E_2, E_3, E_4, E_5, E_6, E_7 and E_8 are the exceptional divisors with the intersection:

We have $-K_S\sim C+E_1+E_2+E_3+E_4+E_5+E_6+E_7+E_8$. Let P be a point on S.

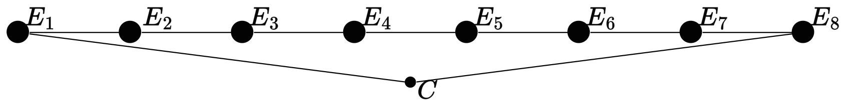


Figure 9.21: Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_8$ singularity

Step 1. Suppose $P \in E_4 \cup E_5$. Without loss of generality we can assume that $P \in E_4$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form the following dual graph:

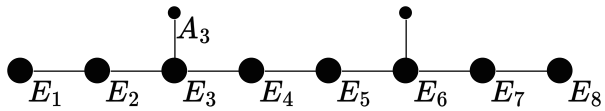


Figure 9.22: Dual graph: $(-K_S)^2 = 1$, $\mathbb{A}_8$ singularity, $\delta_P(S) = 1$

Then $\tau(E_4) = \frac{5}{3}$ and the Zariski Decomposition of the divisor $-K_S - vE_4$ is:

$$P(v) = \begin{cases} -K_S - vE_4 - \frac{v}{20}(5E_1 + 10E_2 + 15E_3 + 16E_5 + 12E_6 + 8E_7 + 4E_8) \text{ if } v \in \big[0, \frac{4}{3}\big], \\ -K_S - vE_4 - (v-1)(E_1 + 2E_2 + 3E_3) - \frac{v}{5}(4E_5 + 3E_6 + 2E_7 + E_8) - (3v-4)A_3 \text{ if } v \in \big[\frac{4}{3}, \frac{5}{3}\big]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{20}(5E_1 + 10E_2 + 15E_3 + 16E_5 + 12E_6 + 8E_7 + 4E_8) \text{ if } v \in \big[0, \frac{4}{3}\big], \\ (v-1)(E_1 + 2E_2 + 3E_3) + \frac{v}{5}(4E_5 + 3E_6 + 2E_7 + E_8) + (3v-4)A_3 \text{ if } v \in \big[\frac{4}{3}, \frac{5}{3}\big]. \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_4 \sim_{\mathbb{R}} \Big(\frac{5}{3} - v\Big)E_4 + \frac{1}{3}\Big(2E_1 + 4E_2 + 6E_3 + 4E_5 + 3E_6 + 2E_7 + E_8 + 3A_3\Big).$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{9v^2}{20} \text{ if } v \in \big[0, \frac{4}{3}\big], \\ \frac{(5-3v)^2}{5} \text{ if } v \in \big[\frac{4}{3}, \frac{5}{3}\big]. \end{cases} \quad P(v) \cdot E_4 = \begin{cases} \frac{9v}{20} \text{ if } v \in \big[0, \frac{4}{3}\big], \\ 3(2 - \frac{3v}{5}) \text{ if } v \in \big[\frac{4}{3}, \frac{5}{3}\big]. \end{cases}$$

We have $S_S(E_4) = 1$. Thus, $\delta_P(S) \le 1$ for $P \in E_4$. Moreover, if $P \in E_4 \cap E_5$ or if $P \in E_4 \backslash E_5$ we have

$$h(v) \le \begin{cases} \frac{369v^2}{800} \text{ if } v \in \big[0, \frac{4}{3}\big], \\ \frac{3(3v-5)(v-15)}{50} \text{ if } v \in \big[\frac{4}{3}, \frac{5}{3}\big]. \end{cases} \quad \text{or } h(v) \le \begin{cases} \frac{351v^2}{800} \text{ if } v \in \big[0, \frac{4}{3}\big], \\ \frac{9(3v-5)(5-7v)}{50} \text{ if } v \in \big[\frac{4}{3}, \frac{5}{3}\big]. \end{cases}$$

Thus, $S(W^{E_2}_{\bullet,\bullet};P) \leq 1$. We get $\delta_P(S) = 1$ for $P \in E_4 \cup E_5$.

Step 2. Suppose $P \in E_3 \cup E_5$. Without loss of generality we can assume that $P \in E_3$ since the proof is similar in other cases. Then $\tau(E_3) = 2$ and the Zariski Decomposition of the divisor $-K_S - vE_3$ is:

$$P(v) = \begin{cases} -K_S - vE_3 - \frac{v}{6}(2E_1+4E_2+5E_4+4E_5+3E_6+2E_7+E_8) \text{ if } v \in [0,1], \\ -K_S - vE_3 - \frac{v}{6}(2E_1+4E_2+5E_4+4E_5+3E_6+2E_7+E_8) - (v-1)A_3 \text{ if } v \in [1,2]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(2E_1+4E_2+5E_4+4E_5+3E_6+2E_7+E_8) \text{ if } v \in [0,1], \\ \frac{v}{6}(2E_1+4E_2+5E_4+4E_5+3E_6+2E_7+E_8) + (v-1)A_3 \text{ if } v \in [1,2]. \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_3 \sim_{\mathbb{R}} (2-v)E_3 + \frac{1}{3}\Big(2E_1+4E_2+5E_4+4E_5+3E_6+2E_7+E_8\Big) + A_3.$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{2} \text{ if } v \in [0,1], \\ \frac{(2-v)^2}{2} \text{ if } v \in [1,2]. \end{cases} \quad P(v)\cdot E_3 = \begin{cases} \frac{v}{2} \text{ if } v \in [0,1], \\ 1 - \frac{v}{2} \text{ if } v \in [1,2]. \end{cases}$$

We have $S_S(E_3) = 1$. Thus, $\delta_P(S) \leq 1$ for $P \in E_3$. Moreover, if $P \in E_4 \cap E_2$ or if $P \in E_4 \backslash (E_2 \cup E_4)$ we have

$$h(v) \leq \begin{cases} \frac{11v^2}{24} \text{ if } v \in [0,1], \\ \frac{(2-v)(5v+6)}{24} \text{ if } v \in [1,2]. \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{v^2}{8} \text{ if } v \in [0,1], \\ \frac{(2-v)(3v-2)}{8} \text{ if } v \in [1,2]. \end{cases}$$

Thus, $S(W^{E_3}_{\bullet,\bullet};P) \leq \frac{5}{6} < 1$ or $S(W^{E_3}_{\bullet,\bullet};P) \leq \frac{1}{3} < 1$. We get $\delta_P(S) = 1$ for $P \in (E_3 \cup E_6)\backslash(E_4 \cup E_5)$.

Step 3. Suppose $P \in E_2 \cup E_7$. Then $\tau(E_2) = \frac{4}{3}$ and the Zariski Decomposition of the divisor $-K_S - vE_2$ is:

$$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{2}E_1 - \frac{v}{7}(6E_3+5E_4+4E_5+3E_6+2E_7+E_7) \text{ if } v \in \big[0,\frac{7}{6}\big], \\ -K_S - vE_2 - \frac{v}{2}E_1 - (v-1)(6E_3+5E_4+4E_5+3E_6+2E_7+E_7) - (6v-7)A_3 \text{ if } v \in \big[\frac{7}{6},\frac{4}{3}\big]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}E_1 + \frac{v}{7}(6E_3+5E_4+4E_5+3E_6+2E_7+E_7) \text{ if } v \in \big[0,\frac{7}{6}\big], \\ \frac{v}{2}E_1 + (v-1)(6E_3+5E_4+4E_5+3E_6+2E_7+E_7) + (6v-7)A_3 \text{ if } v \in \big[\frac{7}{6},\frac{4}{3}\big]. \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_2 \sim_{\mathbb{R}} \Big(\frac{4}{3} - v\Big)E_2 + \frac{1}{3}\Big(2E_1+6E_3+5E_4+4E_5+3E_6+2E_7+E_7+3A_3\Big).$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{9v^2}{14} \text{ if } v \in \left[0, \frac{7}{6}\right], \\ \frac{(4-3v)^2}{2} \text{ if } v \in \left[\frac{7}{6}, \frac{4}{3}\right]. \end{cases} \quad P(v) \cdot E_2 = \begin{cases} \frac{9v}{14} \text{ if } v \in \left[0, \frac{7}{6}\right], \\ 3(1 - \frac{3v}{2}) \text{ if } v \in \left[\frac{7}{6}, \frac{4}{3}\right]. \end{cases}$$

We have $S_S(E_2) = \frac{5}{6}$. Thus, $\delta_P(S) \leq \frac{6}{5}$ for $P \in E_2$. Moreover, if $P \in E_2 \backslash E_3$ we have

$$h(v) \leq \begin{cases} \frac{207v^2}{392} \text{ if } v \in \left[0, \frac{7}{6}\right], \\ \frac{3(3v-4)(7v-12)}{8} \text{ if } v \in \left[\frac{7}{6}, \frac{4}{3}\right]. \end{cases}$$

Thus

$$S(W^{E_2}_{\bullet,\bullet}; P) \leq 2\Big(\int_0^{7/6} \frac{207v^2}{392} dv + \int_{7/6}^{4/3} \frac{3(3v-4)(7v-12)}{8} dv\Big) = \frac{1}{4} < \frac{5}{6}$$

We get $\delta_P(S) = \frac{6}{5}$ for $P \in (E_2 \cup E_7) \backslash (E_3 \cup E_6)$.

Step 4. Suppose $P \in E_1 \cup E_8$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. Then $\tau(E_1) = 1$ and the Zariski decomposition of the divisor $-K_S - vE_1 \sim C + (1-v)E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8$ is given by:

$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{8}(7E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + E_8) \text{ if } v \in \left[0, \frac{8}{9}\right], \\ -K_S - vE_1 - (2v-1)E_2 - (3v-2)E_3 - (4v-3)E_4 - (5v-4)E_5 - (6v-5)E_6 - (7v-6)E_7 - (8v-7)E_8 - (9v-8)C \text{ if } v \in \left[\frac{8}{9}, 1\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{8}(7E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + E_8) \text{ if } v \in \left[0, \frac{8}{9}\right], \\ (2v-1)E_2 + (3v-2)E_3 + (4v-3)E_4 + (5v-4)E_5 + (6v-5)E_6 + (7v-6)E_8 + (9v-8)C \text{ if } v \in \left[\frac{8}{9}, 1\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{9v^2}{8} \text{ if } v \in \left[0, \frac{8}{9}\right], \\ 9(v-1)^2 \text{ if } v \in \left[\frac{8}{9}, 1\right]. \end{cases} \quad P(v) \cdot E_1 = \begin{cases} \frac{9v}{8} \text{ if } v \in \left[0, \frac{8}{9}\right], \\ 9(1-v) \text{ if } v \in \left[\frac{8}{9}, 1\right]. \end{cases}$$

We have $S_S(E_1) = \frac{17}{27}$. Thus, $\delta_P(S) \leq \frac{27}{17}$ for $P \in E_1 \backslash E_2$. Moreover, for such points we have

$$h(v) = \begin{cases} \frac{81v^2}{128} \text{ if } v \in \left[0, \frac{8}{9}\right], \\ \frac{9(1-v)(9v-7)}{2} \text{ if } v \in \left[\frac{8}{9}, 1\right]. \end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet}; P) \leq \frac{10}{27} < \frac{17}{27}$. We get $\delta_P(S) = \frac{27}{17}$ for $P \in (E_1 \cup E_8) \backslash (E_2 \cup E_7)$.
Thus, $\delta_{\mathscr{P}}(X) = 1$. □

$\mathbb{D}_4$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{D}_4$ singularity at point $\mathscr{P}$. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X) = 1$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathscr{C}$ on S and E, E_1, E_2 and E_3 are the exceptional divisors with the intersection:

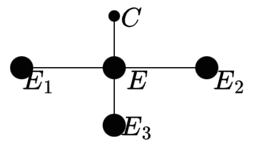


Figure 9.23: Dual graph: $(-K_S)^2 = 1$, $\mathbb{D}_4$ singularity

We have $-K_S \sim C + 2E + E_1 + E_2 + E_3$. Let P be a point on S.

Step 1. Suppose $P \in E$. Then $\tau(E) = 2$ and the Zariski decomposition of the divisor $-K_S - vE \sim (2-v)E + E_1 + E_2 + E_3 + C$ is:

$$P(v) = \begin{cases} -K_S - vE - \frac{v}{2}(E_1+E_2+E_3) \text{ if } v \in [0,1], \\ -K_S - vE - \frac{v}{2}(E_1+E_2+E_3) - (v-1)C \text{ if } v \in [1,2]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(E_1+E_2+E_3) \text{ if } v \in [0,1], \\ \frac{v}{2}(E_1+E_2+E_3) + (v-1)C \text{ if } v \in [1,2]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{2} \text{ if } v \in [0,1], \\ \frac{(2-v)^2}{2} \text{ if } v \in [1,2]. \end{cases} \quad P(v)\cdot E = \begin{cases} \frac{v}{2} \text{ if } v \in [0,1], \\ 1 - \frac{v}{2} \text{ if } v \in [1,2]. \end{cases}$$

We have $S_S(E) = 1$ Thus, $\delta_P(S) \le 1$ for $P \in E$. Moreover, if $P \in E \cap (E_1 \cup E_2 \cup E_3)$ or if $P \in E \backslash (E_1 \cup E_2 \cup E_3)$ we have

$$h(v) \le \begin{cases} \frac{3v^2}{8} \text{ if } v \in [0,1], \\ \frac{(2-v)(2+v)}{24} \text{ if } v \in [1,2]. \end{cases} \quad \text{or } h(v) \le \begin{cases} \frac{v^2}{8} \text{ if } v \in [0,1], \\ \frac{(2-v)(3v-2)}{8} \text{ if } v \in [1,2]. \end{cases}$$

Thus, $S(W^{E}_{\bullet,\bullet};P) \leq \frac{2}{3} < 1$ or $S(W^{E}_{\bullet,\bullet};P) \leq \frac{1}{3} < 1$. We get $\delta_P(S) = 1$ for $P \in E$.

Step 2. Suppose $P \in E_1 \cup E_2 \cup E_3$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. Then $\tau(E_1) = 1$ and the Zariski decomposition of the divisor $-K_S - vE_1 \sim C + 2E + (1-v)E_1 + E_2 + E_3$ is given by:

$$P(v) = -K_S - vE_1 - \frac{v}{2}(2E + E_1 + E_2) \text{ and } N(v) = \frac{v}{2}(2E + E_1 + E_2) \text{ if } v \in [0,1].$$

Moreover,

$$(P(v))^2 = (1-v)(1+v) \text{ and } P(v) \cdot E_1 = v \text{ if } v \in [0,1].$$

We have $S_S(E_1) = \frac{2}{3}$. Thus, $\delta_P(S) \leq \frac{3}{2}$ for $P \in E_1$. Moreover, for $E_1 \backslash E$ such points we have $h(v) \leq \frac{v^2}{2}$ if $v \in [0,1]$. Thus, $S(W^{E_1}_{\bullet,\bullet};P) \leq \frac{1}{3} < \frac{2}{3}$. We get $\delta_P(S) = \frac{3}{2}$ for $P \in (E_1 \cup E_2 \cup E_3) \backslash E$. Thus, $\delta_{\mathscr{P}}(X) = 1$. □

$\mathbb{D}_5$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{D}_5$ singularity at point $\mathscr{P}$. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X) = \frac{6}{7}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathscr{C}$ on S and E, E_1, E_2, E_3 and E_4 are the exceptional divisors with the intersection:

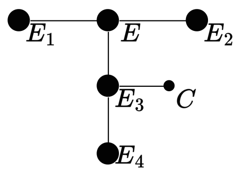


Figure 9.24: Dual graph: $(-K_S)^2 = 1$, $\mathbb{D}_5$ singularity

We have $-K_S \sim C + E_1 + E_2 + 2E + 2E_3 + E_4$. Let P be a point on S.

Step 1. Suppose $P \in E$. Then $\tau(E) = 2$ and the Zariski decomposition of the divisor $-K_S - vE \sim (2-v)E + E_1 + E_2 + 2E_3 + E_4 + C$ is:

$$P(v) = \begin{cases} -K_S - vE - \frac{v}{6}(3E_1 + 3E_2 + 4E_3 + 2E_4) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ -K_S - vE - \frac{v}{2}(E_1 + E_2) - (v-1)(2E_3 + E_4) - (2v-3)C \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(3E_1 + 3E_2 + 4E_3 + 2E_4) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{v}{2}(E_1 + E_2) + (v-1)(2E_3 + E_4) + (2v-3)C \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{3} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ (2-v)^2 \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases} \qquad P(v)\cdot E = \begin{cases} \frac{v}{3} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ 2-v \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

We have $S_S(E) = \frac{7}{6}$. Thus, $\delta_P(S) \le \frac{6}{7}$ for $P \in E$. Moreover, if $P \in E \cap (E_1 \cup E_2)$ or if $P \in E \backslash (E_1 \cup E_2)$ we have

$$h(v) \le \begin{cases} \frac{2v^2}{9} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ 2-v \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases} \quad \text{or } h(v) \le \begin{cases} \frac{5v^2}{18} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{(2-v)(3v-2)}{2} \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

Thus, $S(W^E_{\bullet,\bullet};P) \le \frac{3}{4} < \frac{7}{6}$ or $S(W^E_{\bullet,\bullet};P) \le 1 < \frac{7}{6}$. We get $\delta_P(S) = \frac{6}{7}$ for $P \in E$.

Step 2. Suppose $P \in E_1 \cup E_2$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:

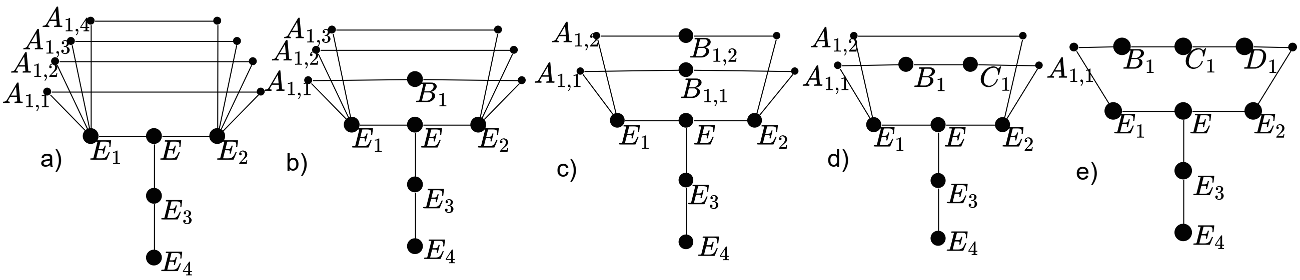


Figure 9.25: Dual graph: $(-K_S)^2 = 1$, $\mathbb{D}_5$ singularity, $\delta_P(S) = \frac{4}{3}$

Then $\tau(E_1) = \frac{5}{4}$ and the Zariski Decomposition of the divisor $-K_S - vE_1$ is:

a).
$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) \text{ if } v \in [0,1], \\ -K_S - vE_1 - \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) - (v-1)(A_{1,1} + A_{1,2} + A_{1,3} + A_{1,4}) \text{ if } v \in \left[1, \frac{5}{4}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) \text{ if } v \in [0,1], \\ \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) + (v-1)(A_{1,1} + A_{1,2} + A_{1,3} + A_{1,4}) \text{ if } v \in \left[1, \frac{5}{4}\right]. \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) \text{ if } v \in [0,1], \\ -K_S - vE_1 - \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) - (v-1)(2A_{1,1} + B_1 + A_{1,2} + A_{1,3}) \text{ if } v \in \left[1, \frac{5}{4}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) \text{ if } v \in [0,1], \\ \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) + (v-1)(2A_{1,1} + B_1 + A_{1,2} + A_{1,3}) \text{ if } v \in \left[1, \frac{5}{4}\right]. \end{cases}$$

c).
$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) \text{ if } v \in [0,1], \\ -K_S - vE_1 - \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) - (v-1)(2A_{1,1} + B_{1,1} + A_{1,2} + B_{1,2}) \text{ if } v \in \left[1, \frac{5}{4}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(6E+3E_2+4E_3+2E_4) \text{ if } v \in [0,1], \\ \frac{v}{5}(6E+3E_2+4E_3+2E_4)+(v-1)(2A_{1,1}+B_{1,1}+A_{1,2}+B_{1,2}) \text{ if } v \in \left[1,\frac{5}{4}\right]. \end{cases}$$

d). $$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{5}(6E+3E_2+4E_3+2E_4) \text{ if } v \in [0,1], \\ -K_S - vE_1 - \frac{v}{5}(6E+3E_2+4E_3+2E_4)-(v-1)(3A_{1,1}+2B_1+C_1+A_{1,2}) \text{ if } v \in \left[1,\frac{5}{4}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(6E+3E_2+4E_3+2E_4) \text{ if } v \in [0,1], \\ \frac{v}{5}(6E+3E_2+4E_3+2E_4)+(v-1)(3A_{1,1}+2B_1+C_1+A_{1,2}) \text{ if } v \in \left[1,\frac{5}{4}\right]. \end{cases}$$

e). $$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{5}(6E+3E_2+4E_3+2E_4) \text{ if } v \in [0,1], \\ -K_S - vE_1 - \frac{v}{5}(6E+3E_2+4E_3+2E_4)-(v-1)(4A_{1,1}+3B_1+2C_1+D_1) \text{ if } v \in \left[1,\frac{5}{4}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(6E+3E_2+4E_3+2E_4) \text{ if } v \in [0,1], \\ \frac{v}{5}(6E+3E_2+4E_3+2E_4)+(v-1)(4A_{1,1}+3B_1+2C_1+D_1) \text{ if } v \in \left[1,\frac{5}{4}\right]. \end{cases}$$

The Zariski Decomposition in part a). follows from

$$-K_S - vE_1 \sim_{\mathbb{R}} \left(\frac{5}{4}-v\right)E_1 + \frac{1}{4}\Big(6E+3E_2+4E_3+2E_4+A_{1,1}+A_{1,2}+A_{1,3}+A_{1,4}\Big).$$

A similar statement holds in other parts. Moreover,

$$(P(v))^2 = \begin{cases} 1-\frac{4v^2}{5} \text{ if } v \in [0,1], \\ \frac{(5-4v)^2}{5} \text{ if } v \in \left[1,\frac{5}{4}\right]. \end{cases} \quad P(v)\cdot E_1 = \begin{cases} \frac{4v}{5} \text{ if } v \in [0,1], \\ 4(1-\frac{4v}{5}) \text{ if } v \in \left[1,\frac{5}{4}\right]. \end{cases}$$

We have $S_S(E_1) = \frac{3}{4}$. Thus, $\delta_P(S) \le \frac{4}{3}$ for $P \in E_1$. Moreover, if $P \in E_1 \backslash E$ we have

$$h(v) \le \begin{cases} \frac{8v^2}{25} \text{ if } v \in [0,1], \\ \frac{4(5-4v)(7v-5)}{25} \text{ if } v \in \left[1,\frac{5}{4}\right]. \end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet};P) \le \frac{19}{60} < \frac{3}{4}$. We get $\delta_P(S) = \frac{4}{3}$ for $P \in (E_1 \cup E_2)\backslash E$.

Step 3. Suppose $P \in E_3$. Then $\tau(E_3) = 2$ and the Zariski decomposition of the divisor $-K_S - vE_3 \sim C+E_1+E_2+2E+(2-v)E_3+E_4$ is:

$$P(v) = \begin{cases} -K_S - vE_3 - \frac{v}{2}(E_4+2E+E_1+E_2) \text{ if } v \in [0,1], \\ -K_S - vE_3 - \frac{v}{2}(E_4+2E+E_1+E_2)-(v-1)C \text{ if } v \in [1,2]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(E_4+2E+E_1+E_2) \text{ if } v \in [0,1], \\ \frac{v}{2}(E_4+2E+E_1+E_2)+(v-1)C \text{ if } v \in [1,2]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1-\frac{v^2}{2} \text{ if } v\in[0,1], \\ \frac{(2-v)^2}{2} \text{ if } v\in[1,2]. \end{cases} \quad P(v)\cdot E_3 = \begin{cases} \frac{v}{2} \text{ if } v\in[0,1], \\ 1-\frac{v}{2} \text{ if } v\in[1,2]. \end{cases}$$

Now we apply the computation from Section 9.1 (Step 1.) and get that $\delta_P(S)=1$ for $P\in E_3\backslash E$.

Step 4. Suppose $P\in E_4$. Then $\tau(E_4)=1$ and the Zariski decomposition of the divisor $-K_S-vE_4\sim C+E_1+E_2+2E+2E_3+(1-v)E_4$ is given by:

$$P(v)=-K_S-vE_4-\frac{v}{2}(2E_3+2E+E_1+E_2) \text{ and } N(v)=\frac{v}{2}(2E_3+2E+E_1+E_2) \text{ if } v\in[0,1].$$

Moreover,

$$(P(v))^2=(1-v)(1+v) \text{ and } P(v)\cdot E_4=v \text{ if } v\in[0,1].$$

Now we apply the computation from Section 9.1 (Step 2.) and get that $\delta_P(S)=\frac{3}{2}$ for $P\in E_4\backslash E_3$. Thus, $\delta_{\mathscr{P}}(X)=\frac{6}{7}$. □

$\mathbb{D}_6$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{D}_6$ singularity at point $\mathscr{P}$. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X)=\frac{3}{4}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathscr{C}$ on S and E, E_1, E_2, E_3, E_4 and E_5 are the exceptional divisors with the intersection:

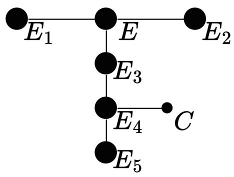


Figure 9.26: Dual graph: $(-K_S)^2=1$, $\mathbb{D}_6$ singularity

We have $-K_S\sim C+E_1+E_2+2E+2E_3+2E_4+E_5$. Let P be a point on S.

Step 1. Suppose $P\in E$. Then $\tau(E)=2$ and the Zariski decomposition of the divisor $-K_S-vE\sim C+E_1+E_2+(2-v)E+2E_3+2E_4+E_5$ is given by:

$$P(v)=-K_S-vE-\frac{v}{4}(2E_1+2E_2+3E_3+2E_4+E_5) \text{ if } v\in[0,2].$$
$$N(v)=\frac{v}{4}(2E_1+2E_2+3E_3+2E_4+E_5) \text{ if } v\in[0,2].$$

Moreover,

$$(P(v))^2=\frac{(2-v)(2+v)}{4}P(v)\cdot E=\frac{v}{4} \text{ and } \text{ if } v\in[0,2].$$

We have $S_S(E)=\frac{4}{3}$. Thus, $\delta_P(S)\le\frac{3}{4}$ for $P\in E$. Moreover, for such points we have $h(v)\le\frac{7v^2}{32}$ if $v\in[0,2]$. Thus, $S(W^E_{\bullet,\bullet};P)\le\frac{7}{6}<\frac{4}{3}$. We get $\delta_P(S)=\frac{3}{4}$ for $P\in E$.

Step 2. Suppose $P\in E_1\cup E_2$. Without loss of generality we can assume that $P\in E_1$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:

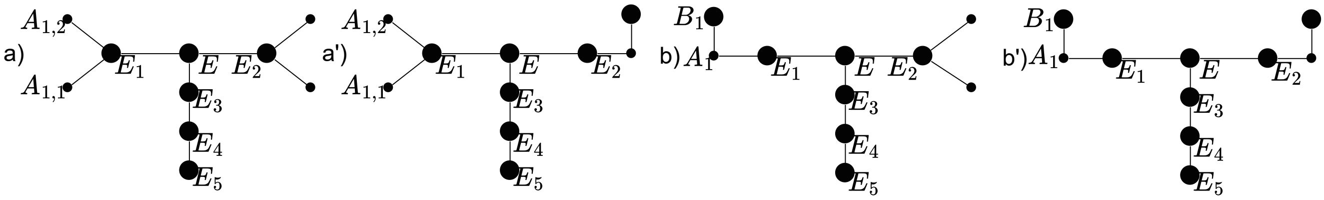


Figure 9.27: Dual graph: $(-K_S)^2=1$, $\mathbb{D}_6$ singularity, $\delta_P(S)=\frac{6}{5}$

Then $\tau(E_1)=\frac{3}{2}$ and the Zariski Decomposition of the divisor $-K_S-vE_1$ is:

a, a'). $$P(v)=\begin{cases}-K_S-vE_1-\frac{v}{3}(4E+2E_2+3E_3+2E_4+E_5) \text{ if } v\in[0,1],\\ -K_S-vE_1-\frac{v}{3}(4E+2E_2+3E_3+2E_4+E_5)-(v-1)(A_{1,1}+A_{1,2}) \text{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{3}(4E+2E_2+3E_3+2E_4+E_5) \text{ if } v\in[0,1],\\ \frac{v}{3}(4E+2E_2+3E_3+2E_4+E_5)+(v-1)(A_{1,1}+A_{1,2}) \text{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

b, b'). $$P(v)=\begin{cases}-K_S-vE_1-\frac{v}{3}(4E+2E_2+3E_3+2E_4+E_5) \text{ if } v\in[0,1],\\ -K_S-vE_1-\frac{v}{3}(4E+2E_2+3E_3+2E_4+E_5)-(v-1)(2A_{1,1}+B_1) \text{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{3}(4E+2E_2+3E_3+2E_4+E_5) \text{ if } v\in[0,1],\\ \frac{v}{3}(4E+2E_2+3E_3+2E_4+E_5)+(v-1)(2A_{1,1}+B_1) \text{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

The Zariski Decomposition in part a). follows from

$$-K_S-vE_1\sim_{\mathbb{R}}\Big(\frac{3}{2}-v\Big)E_1+\frac{1}{2}\Big(4E+2E_2+3E_3+2E_4+E_5+A_{1,1}+A_{1,2}\Big).$$

A similar statement holds in other parts. Moreover,

$$(P(v))^2=\begin{cases}1-\frac{2v^2}{3} \text{ if } v\in[0,1],\\ \frac{(3-2v)^2}{3} \text{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}\qquad P(v)\cdot E_1=\begin{cases}\frac{2v}{3} \text{ if } v\in[0,1],\\ 2(1-\frac{2v}{3}) \text{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

We have $S_S(E_1)=\frac{5}{6}$. Thus, $\delta_P(S)\le\frac{6}{5}$ for $P\in E_1$. Moreover, if $P\in E_1\backslash E$ we have

$$h(v)\le\begin{cases}\frac{2v^2}{9} \text{ if } v\in[0,1],\\ \frac{2(2v-3)(4v-3)}{9} \text{ if } v\in\left[1,\frac{3}{2}\right].\end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet};P) \leq \frac{1}{3} < \frac{5}{6}$. We get $\delta_P(S) = \frac{6}{5}$ for $P \in (E_1 \cup E_2)\backslash E$.

Step 3. Suppose $P \in E_3$. Then $\tau(E_3) = 2$ and the Zariski decomposition of the divisor Zariski decomposition of the divisor $-K_S - vE_3 \sim C + E_1 + E_2 + 2E + (2-v)E_3 + 2E_4 + E_5$ is:

$$P(v) = \begin{cases} -K_S - vE_3 - \frac{v}{6}(3E_1 + 3E_2 + 6E + 4E_4 + 2E_5) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ -K_S - vE_3 - \frac{v}{2}(2E + E_1 + E_2) - (v-1)(2E_4 + E_5) - (2v-3)C \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(3E_1 + 3E_2 + 6E + 4E_4 + 2E_5) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{v}{2}(2E + E_1 + E_2) + (v-1)(2E_4 + E_5) + (2v-3)C \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{3} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ (2-v)^2 \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases} \quad P(v) \cdot E_3 = \begin{cases} \frac{v}{3} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ 2 - v \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

Now we apply the computation from Section 9.1 (Step 1.) and get that $\delta_P(S) = \frac{6}{7}$ for $P \in E_3\backslash E$.

Step 4. Suppose $P \in E_4$. Then $\tau(E_4) = 2$ and the Zariski decomposition of the divisor Zariski decomposition of the divisor $-K_S - vE_4 \sim C + E_1 + E_2 + 2E + 2E_3 + (2-v)E_4 + E_5$ is:

$$P(v) = \begin{cases} -K_S - vE_4 - \frac{v}{2}(2E_3 + 2E + E_1 + E_2 + E_5) \text{ if } v \in [0, 1], \\ -K_S - vE_4 - \frac{v}{2}(2E_3 + 2E + E_1 + E_2 + E_5) - (v-1)C \text{ if } v \in [1, 2]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(2E_3 + 2E + E_1 + E_2 + E_5) \text{ if } v \in [0, 1], \\ \frac{v}{2}(2E_3 + 2E + E_1 + E_2 + E_5) + (v-1)C \text{ if } v \in [1, 2]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{2} \text{ if } v \in [0, 1], \\ \frac{(2-v)^2}{2} \text{ if } v \in [1, 2]. \end{cases} \quad P(v) \cdot E_4 = \begin{cases} \frac{v}{2} \text{ if } v \in [0, 1], \\ 1 - \frac{v}{2} \text{ if } v \in [1, 2]. \end{cases}$$

Now we apply the computation from Section 9.1 (Step 1.) and get $\delta_P(S) = 1$ for $P \in E_4\backslash E_3$.

Step 5. Suppose $P \in E_5$. Then $\tau(E_5) = 1$ and the Zariski decomposition of the divisor Zariski decomposition of the divisor $-K_S - vE_5 \sim C + E_1 + E_2 + 2E + 2E_3 + 2E_4 + (1-v)E_5$ is given by:

$$P(v) = -K_S - vE_5 - \frac{v}{2}(2E_4 + 2E_3 + 2E + E_1 + E_2) \text{ if } v \in [0, 1].$$
$$N(v) = \frac{v}{2}(2E_4 + 2E_3 + 2E + E_1 + E_2) \text{ if } v \in [0, 1].$$

Moreover,

$$(P(v))^2 = (1-v)(1+v) \text{ and } P(v) \cdot E_5 = v \text{ if } v \in [0, 1].$$

Now we apply the computation from Section 9.1 (Step 2.) and get that $\delta_P(S) = \frac{3}{2}$ for $P \in E_5 \backslash E_4$. Thus, $\delta_{\mathscr{P}}(X) = \frac{3}{4}$. □

$\mathbb{D}_7$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{D}_7$ singularity at point $\mathscr{P}$. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X) = \frac{2}{3}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathscr{C}$ on S and E, E_1, E_2, E_3, E_4, E_5 and E_6 are the exceptional divisors with the intersection:

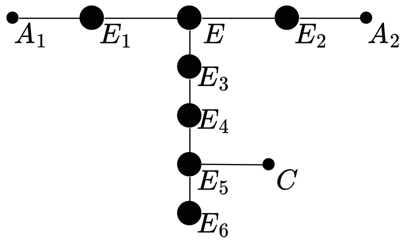


Figure 9.28: Dual graph: $(-K_S)^2 = 1$, $\mathbb{D}_7$ singularity

We have $-K_S \sim C + E_1 + E_2 + 2E + 2E_3 + 2E_4 + 2E_5 + E_6$. Let P be a point on S.
Step 1. Suppose $P \in E$. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves A_1 and A_2 which form the dual graph above with the rest of the curves. Then $\tau(E) = \frac{5}{2}$ and the Zariski decomposition of the divisor Zariski Decomposition of the divisor $-K_S - vE$ is:

$$P(v) = \begin{cases} -K_S - vE - \frac{v}{2}(E_1 + E_2) - \frac{v}{5}(4E_3 + 3E_4 + 2E_5 + E_6) \text{ if } v \in [0,2], \\ -K_S - vE - (v-1)(E_1 + E_2) - \frac{v}{5}(4E_3 + 3E_4 + 2E_5 + E_6) - (v-2)(A_1 + A_2) \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(E_1 + E_2) + \frac{v}{5}(4E_3 + 3E_4 + 2E_5 + E_6) \text{ if } v \in [0,2], \\ (v-1)(E_1 + E_2) + \frac{v}{5}(4E_3 + 3E_4 + 2E_5 + E_6) + (v-2)(A_1 + A_2) \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE \sim_{\mathbb{R}} \Big(\frac{5}{2} - v\Big)E + \frac{1}{2}\Big(3E_1 + 3E_2 + 4E_3 + 3E_4 + 2E_5 + E_6 + A_1 + A_2\Big).$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{5} \text{ if } v \in [0,2], \\ \frac{(5-2v)^2}{5} \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases} \quad P(v) \cdot E = \begin{cases} \frac{v}{5} \text{ if } v \in [0,2], \\ 2(1 - \frac{2v}{5}) \text{ if } v \in \left[2, \frac{5}{2}\right]. \end{cases}$$

We have $S_S(E) = \frac{3}{2}$. Thus, $\delta_P(S) \leq \frac{2}{3}$ for $P \in E$. Moreover, if $P \in E \cap E_3$ if $P \in E \backslash E_3$ we have

$$h(v) \leq \begin{cases} \frac{9v^2}{50} \text{ if } v \in \big[0,2\big], \\ \frac{2(5-2v)(2v+5)}{25} \text{ if } v \in \big[2,\frac{5}{2}\big]. \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{3v^2}{25} \text{ if } v \in \big[0,2\big], \\ \frac{6v(5-2v)}{25} \text{ if } v \in \big[2,\frac{5}{2}\big]. \end{cases}$$

Thus $S(W^E_{\bullet,\bullet};P) \leq \frac{4}{3} < \frac{3}{2}$ or $S(W^E_{\bullet,\bullet};P) \leq \frac{9}{10} < \frac{3}{2}$. We get $\delta_P(S) = \frac{2}{3}$ for $P \in E$.

Step 2. Suppose $P \in E_1 \cup E_2$. ithout loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. Then $\tau(E_1) = \frac{3}{2}$ and the Zariski decomposition of the divisor Zariski decomposition of the divisor $-K_S - vE_3$ is the following:

$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{7}(10E+5E_2+8E_3+6E_4+4E_5+2E_6) \text{ if } v \in [0,1], \\ -K_S - vE_1 - \frac{v}{7}(10E+5E_2+8E_3+6E_4+4E_5+2E_6) - (v-1)A_1 \text{ if } v \in \big[1,\frac{7}{5}\big], \\ -K_S - vE_1 - (v-1)(10E+8E_3+6E_4+4E_5+2E_6+A_1) - (5v-6)E_2 - (5v-7)A_2 \text{ if } v \in \big[\frac{7}{5},\frac{3}{2}\big]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{7}(10E+5E_2+8E_3+6E_4+4E_5+2E_6) \text{ if } v \in [0,1], \\ \frac{v}{7}(10E+5E_2+8E_3+6E_4+4E_5+2E_6) + (v-1)A_1 \text{ if } v \in \big[1,\frac{7}{5}\big], \\ (v-1)(10E+8E_3+6E_4+4E_5+2E_6+A_1) + (5v-6)E_2 + (5v-7)A_2 \text{ if } v \in \big[\frac{7}{5},\frac{3}{2}\big]. \end{cases}$$

Then $\tau(E_1) = \frac{3}{2}$ and the Zariski Decomposition follows from

$$-K_S - vE_1 \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)E_1 + \frac{1}{2}\Big(2A_2 + 3E_2 + 5E + 4E_3 + 3E_4 + 2E_5 + E_6 + A_1\Big).$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{4v^2}{7} \text{ if } v \in [0,1], \\ 2 - 2v + \frac{3v^2}{7} \text{ if } v \in \big[1,\frac{7}{5}\big], \\ (3-2v)^2 \text{ if } v \in \big[\frac{7}{5},\frac{3}{2}\big]. \end{cases} \quad P(v) \cdot E_1 = \begin{cases} \frac{4v}{7} \text{ if } v \in [0,1], \\ 1 - \frac{3v}{7} \text{ if } v \in \big[1,\frac{7}{5}\big], \\ 2(3-2v) \text{ if } v \in \big[\frac{7}{5},\frac{3}{2}\big]. \end{cases}$$

We have $S_S(E_3) = \frac{9}{10}$. Thus, $\delta_P(S) \leq \frac{10}{9}$ for $P \in E_1$. Moreover, if $P \in E_1 \backslash E$ we have

$$h(v) \leq \begin{cases} \frac{8v^2}{49} \text{ if } v \in [0,1], \\ \frac{(7-3v)(11v-7)}{98} \text{ if } v \in \big[1,\frac{7}{5}\big], \\ 2(3-2v)(2-v) \text{ if } v \in \big[\frac{7}{5},\frac{3}{2}\big]. \end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet};P) \leq \frac{3}{10} < \frac{9}{10}$. We get $\delta_P(S) = \frac{10}{9}$ for $P \in (E_1 \cup E_2) \backslash E$.

Step 3. Suppose $P \in E_3$. Then $\tau(E_3) = 2$ and the Zariski decomposition of the divisor $-K_S - vE_3 \sim C + E_1 + E_2 + 2E + (2-v)E_3 + 2E_4 + 2E_5 + E_6$ is given by:

$$P(v) = -K_S - vE_3 - \frac{v}{4}(2E_1 + 2E_2 + 4E + 3E_4 + 2E_5 + E_6) \text{ if } v \in [0,2].$$

$$N(v) = \frac{v}{4}(2E_1 + 2E_2 + 4E + 3E_4 + 2E_5 + E_6) \text{ if } v \in [0,2].$$

Moreover,

$$(P(v))^2 = \frac{(2-v)(2+v)}{4} \text{ and } P(v)\cdot E_3 = \frac{v}{4} \text{ if } v \in [0,2].$$

Now we apply the computation from Section 9.1 (Step 1.) and get that $\delta_P(S) = \frac{3}{4}$ for $P \in E_3 \backslash E$.

Step 4. Suppose $P \in E_4$. Then $\tau(E_4) = 2$ and the Zariski decomposition of the divisor $-K_S - vE_4 \sim C + E_1 + E_2 + 2E + 2E_3 + (2-v)E_4 + 2E_5 + E_6$ is:

$$P(v) = \begin{cases} -K_S - vE_4 - \frac{v}{6}(3E_1 + 3E_2 + 6E + 6E_3 + 4E_5 + 2E_6) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ -K_S - vE_4 - \frac{v}{2}(E_1 + E_2 + 2E + 2E_3) - (v-1)(2E_5 + E_6) - (2v-3)C \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(3E_1 + 3E_2 + 6E + 6E_3 + 4E_5 + 2E_6) \text{ if } v \in \left[0, \frac{3}{2}\right], \\ \frac{v}{2}(E_1 + E_2 + 2E + 2E_3) + (v-1)(2E_5 + E_6) + (2v-3)C \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{3} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ (2-v)^2 \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases} \quad P(v)\cdot E_4 = \begin{cases} \frac{v}{3} \text{ if } v \in \left[0, \frac{3}{2}\right], \\ 2 - v \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

Now we apply the computation from Section 9.1 (Step 1.) and get that $\delta_P(S) = \frac{6}{7}$ for $P \in E_4 \backslash E_3$.

Step 5. Suppose $P \in E_5$. Then $\tau(E_5) = 2$ and the Zariski decomposition of the divisor $-K_S - vE_5 \sim C + E_1 + E_2 + 2E + 2E_3 + 2E_4 + (2-v)E_5 + E_6$ is:

$$P(v) = \begin{cases} -K_S - vE_5 - \frac{v}{2}(2E_4 + 2E_3 + 2E + E_1 + E_2 + E_6) \text{ if } v \in [0,1], \\ -K_S - vE_5 - \frac{v}{2}(2E_4 + 2E_3 + 2E + E_1 + E_2 + E_6) - (v-1)C \text{ if } v \in [1,2]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(2E_4 + 2E_3 + 2E + E_1 + E_2 + E_6) \text{ if } v \in [0,1], \\ \frac{v}{2}(2E_4 + 2E_3 + 2E + E_1 + E_2 + E_6) + (v-1)C \text{ if } v \in [1,2]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{2} \text{ if } v \in [0,1], \\ \frac{(2-v)^2}{2} \text{ if } v \in [1,2]. \end{cases} \quad P(v)\cdot E_5 = \begin{cases} \frac{v}{2} \text{ if } v \in [0,1], \\ 1 - \frac{v}{2} \text{ if } v \in [1,2]. \end{cases}$$

Now we apply the computation from Section 9.1 (Step 1.) and get that $\delta_P(S) = 1$ for $P \in E_5 \backslash E_4$.

Step 6. Suppose $P \in E_6$. Then $\tau(E_6) = 1$ and the Zariski decomposition of the divisor $-K_S - vE_6 \sim C + E_1 + E_2 + 2E + 2E_3 + 2E_4 + 2E_5 + (1-v)E_6$ is given by:

$$P(v) = -K_S - vE_6 - \frac{v}{2}(2E_5 + 2E_4 + 2E_3 + 2E + E_1 + E_2) \text{ if } v \in [0,1].$$

$$N(v) = \frac{v}{2}(2E_5 + 2E_4 + 2E_3 + 2E + E_1 + E_2) \text{ if } v \in [0,1].$$

Moreover,

$$(P(v))^2 = (1-v)(1+v) \text{ and } P(v) \cdot E_6 = v \text{ if } v \in [0,1].$$

Now we apply the computation from Section 9.1 (Step 2.) and get that $\delta_P(S) = \frac{3}{2}$ for $P \in E_6 \backslash E_5$. Thus, $\delta_{\mathscr{P}}(X) = \frac{2}{3}$. □

$\mathbb{D}_8$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{D}_8$ singularity at point $\mathscr{P}$. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X) = \frac{3}{5}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathscr{C}$ on S and E, E_1, E_2, E_3, E_4, E_5, E_6 and E_7 are the exceptional divisors with the intersection:

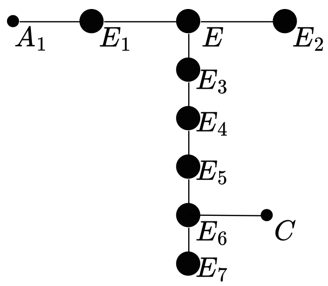


Figure 9.29: Dual graph: $(-K_S)^2 = 1$, $\mathbb{D}_8$ singularity

We have $-K_S \sim C + E_1 + E_2 + 2E + 2E_3 + 2E_4 + 2E_5 + 2E_6 + E_7$. Let P be a point on S.

Step 1. Suppose $P \in E$.If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist a (-1)-curve A_1 which form the dual graph above with the rest of the curves. Then the corresponding Zariski Decomposition of the divisor $-K_S - vE$ is:

$$P(v) = \begin{cases} -K_S - vE - \frac{v}{6}(3E_1 + 3E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in [0,2], \\ -K_S - vE - (v-1)E_1 - \frac{v}{6}(3E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) - (v-2)A_1 \text{ if } v \in [2,3]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(3E_1 + 3E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in [0,2], \\ (v-1)E_1 + \frac{v}{6}(3E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) + (v-2)A_1 \text{ if } v \in [2,3]. \end{cases}$$

Then $\tau(E) = 3$ and the Zariski Decomposition follows from

$$-K_S - vE \sim_{\mathbb{R}} (3-v)E + \frac{1}{2}\Big(4E_1 + 3E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7 + 2A_1\Big).$$

Moreover,

$$(P(v))^2 = \begin{cases} 1-\frac{v^2}{6} \text{ if } v \in [0,2], \\ \frac{(3-v)^2}{3} \text{ if } v \in [2,3]. \end{cases} \quad P(v)\cdot E = \begin{cases} \frac{v}{6} \text{ if } v \in [0,2], \\ 1-\frac{v}{6} \text{ if } v \in [2,3]. \end{cases}$$

We have $S_S(E) = \frac{5}{3}$. Thus, $\delta_P(S) \leq \frac{3}{5}$ for $P \in E$. Moreover, if $P \in E \cap E_1$ if $P \in E \backslash E_1$ we have

$$h(v) \leq \begin{cases} \frac{7v^2}{72} \text{ if } v \in \big[0,2\big], \\ \frac{(3-v)(5v-3)}{18} \text{ if } v \in [2,3]. \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{11v^2}{72} \text{ if } v \in \big[0,2\big], \\ \frac{(3-v)(4v+3)}{18} \text{ if } v \in [2,3]. \end{cases}$$

Thus, $S(W^E_{\bullet,\bullet};P) \leq 1 < \frac{5}{3}$ or $S(W^E_{\bullet,\bullet};P) \leq \frac{3}{2} < \frac{5}{3}$. We get $\delta_P(S) = \frac{3}{5}$ for $P \in E$.

Step 2. Suppose $P \in E_1$. Then $\tau(E_1) = 2$ and the Zariski decomposition of the divisor $-K_S - vE_1$ is:

$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{4}(6E+3E_2+5E_3+4E_4+3E_5+2E_6+E_7) \text{ if } v \in [0,1], \\ -K_S - vE_1 - \frac{v}{4}(6E+3E_2+5E_3+4E_4+3E_5+2E_6+E_7) - (v-1)A_1 \text{ if } v \in [1,2]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(6E+3E_2+5E_3+4E_4+3E_5+2E_6+E_7) \text{ if } v \in [0,1], \\ \frac{v}{4}(6E+3E_2+5E_3+4E_4+3E_5+2E_6+E_7) + (v-1)A_1 \text{ if } v \in [1,2]. \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_1 \sim_{\mathbb{R}} (2-v)E_1 + \frac{1}{2}\Big(6E+3E_2+5E_3+4E_4+3E_5+2E_6+E_7+2A_1\Big).$$

Moreover,

$$(P(v))^2 = \begin{cases} 1-\frac{v^2}{2} \text{ if } v \in [0,1], \\ \frac{(2-v)^2}{2} \text{ if } v \in [1,2]. \end{cases} \quad P(v)\cdot E_1 = \begin{cases} \frac{v}{2} \text{ if } v \in [0,1], \\ 1-\frac{v}{2} \text{ if } v \in [1,2]. \end{cases}$$

Now we apply the computation from Section 9.1 (Step 1.) and get $\delta_P(S) = 1$ for $P \in E_1 \backslash E$.

Step 3. Suppose $P \in E_2$. Then $\tau(E_2) = \frac{3}{2}$ and the Zariski decomposition of the divisor $-K_S - vE_2$ is the following:

$$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{4}(6E+3E_1+5E_3+4E_4+3E_5+2E_6+E_7) \text{ if } v \in \big[0,\frac{4}{3}\big], \\ -K_S - vE_2 - (v-1)(6E+5E_3+4E_4+3E_5+2E_6+E_7) - (6v-7)E_1 - (6v-8)A_1 \text{ if } v \in \big[\frac{4}{3},\frac{3}{2}\big]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(6E+3E_1+5E_3+4E_4+3E_5+2E_6+E_7) \text{ if } v \in \big[0,\frac{4}{3}\big], \\ (v-1)(6E+5E_3+4E_4+3E_5+2E_6+E_7) + (6v-7)E_1 + (6v-8)A_1 \text{ if } v \in \big[\frac{4}{3},\frac{3}{2}\big]. \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_2 \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)E_2 + \frac{1}{2}\Big(6E + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7 + 4E_1 + 2A_1\Big).$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{2} \text{ if } v \in \big[0, \frac{4}{3}\big], \\ (3-2v)^2 \text{ if } v \in \big[\frac{4}{3}, \frac{3}{2}\big]. \end{cases} \quad P(v) \cdot E_2 = \begin{cases} \frac{v}{2} \text{ if } v \in \big[0, \frac{4}{3}\big], \\ 2(3-2v) \text{ if } v \in \big[\frac{4}{3}, \frac{3}{2}\big]. \end{cases}$$

We have $S_S(E_2) = \frac{17}{18}$. Thus, $\delta_P(S) \leq \frac{18}{17}$ for $P \in E_2$. Moreover, if $P \in E_2 \backslash E_1$ we have:

$$h(v) \leq \begin{cases} \frac{v^2}{8} \text{ if } v \in \big[0, \frac{4}{3}\big], \\ 2(3-2v)^2 \text{ if } v \in \big[\frac{4}{3}, \frac{3}{2}\big]. \end{cases}$$

Thus, $S(W^{E_2}_{\bullet,\bullet}; P) \leq \frac{2}{9} < \frac{17}{18}$. We get $\delta_P(S) = \frac{18}{17}$ for $P \in E_2 \backslash E_1$.

Step 4. Suppose $P \in E_3$. Then $\tau(E_3) = \frac{5}{2}$ and the Zariski decomposition of the divisor $-K_S - vE_3$ is the following:

$$P(v) = \begin{cases} -K_S - vE_3 - \frac{v}{2}(2E + E_1 + E_2) - \frac{v}{5}(4E_3 + 3E_4 + 2E_5 + E_6) \text{ if } v \in [0, 2], \\ -K_S - vE_3 - (v-1)(2E + E_2) - \frac{v}{5}(4E_3 + 3E_4 + 2E_5 + E_6) - (2v-3)E_1 - (2v-4)A_1 \text{ if } v \in \big[2, \frac{5}{2}\big]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(2E + E_1 + E_2) + \frac{v}{5}(4E_3 + 3E_4 + 2E_5 + E_6) \text{ if } v \in [0, 2], \\ (v-1)(2E + E_2) + \frac{v}{5}(4E_3 + 3E_4 + 2E_5 + E_6) + (2v-3)E_1 + (2v-4)A_1 \text{ if } v \in \big[2, \frac{5}{2}\big]. \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_3 \sim_{\mathbb{R}} \Big(\frac{5}{2} - v\Big)E_3 + \frac{1}{2}\Big(6E + 3E_2 + 4E_3 + 3E_4 + 2E_5 + E_6 + 4E_1 + 2A_1\Big).$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{5} \text{ if } v \in [0, 2], \\ \frac{(5-2v)^2}{5} \text{ if } v \in \big[2, \frac{5}{2}\big]. \end{cases} \quad P(v) \cdot E_3 = \begin{cases} \frac{v}{5} \text{ if } v \in [0, 2], \\ 2(1 - \frac{2v}{5}) \text{ if } v \in \big[2, \frac{5}{2}\big]. \end{cases}$$

Now we apply the computation from Section 9.1 (Step 1.) and get that $\delta_P(S) = \frac{2}{3}$ for $P \in E_3 \backslash E$.

Step 5. Suppose $P \in E_4$. Then $\tau(E_4) = 2$ and the Zariski decomposition of the divisor $-K_S - vE_4 \sim C + E_1 + E_2 + 2E + 2E_3 + (2-v)E_4 + 2E_5 + 2E_6 + E_7$ is given by:

$$P(v) = -K_S - vE_4 - \frac{v}{4}(2E_1 + 2E_2 + 4E + 4E_3 + 3E_4 + 2E_5 + E_6) \text{ if } v \in [0, 2].$$

$$N(v) = \frac{v}{4}(2E_1 + 2E_2 + 4E + 4E_3 + 3E_4 + 2E_5 + E_6) \text{ if } v \in [0, 2].$$

Moreover,

$$(P(v))^2 = \frac{(2-v)(2+v)}{4} \text{ and } P(v)\cdot E_4 = \frac{v}{4} \text{ if } v\in[0,2].$$

Now we apply the computation from Section 9.1 (Step 1.) and get that $\delta_P(S)=\frac{3}{4}$ for $P\in E_4\backslash E_3$.

Step 6. Suppose $P\in E_5$. Then $\tau(E_5)=2$ and the Zariski decomposition of the divisor $-K_S-vE_5\sim C+E_1+E_2+2E+2E_3+2E_4+(2-v)E_5+2E_6+E_7$ is:

$$P(v)=\begin{cases}-K_S-vE_5-\frac{v}{6}(3E_1+3E_2+6E+6E_3+6E_4+4E_6+2E_7) \text{ if } v\in\left[0,\frac{3}{2}\right],\\ -K_S-vE_5-\frac{v}{2}(E_1+E_2+2E+2E_3+2E_4)-(v-1)(2E_6+E_7)-(2v-3)C \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{6}(3E_1+3E_2+6E+6E_3+6E_4+4E_6+2E_7) \text{ if } v\in\left[0,\frac{3}{2}\right],\\ \frac{v}{2}(E_1+E_2+2E+2E_3+2E_4)+(v-1)(2E_6+E_7)+(2v-3)C \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{v^2}{3} \text{ if } v\in\left[0,\frac{3}{2}\right],\\ (2-v)^2 \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}\qquad P(v)\cdot E_5=\begin{cases}\frac{v}{3} \text{ if } v\in\left[0,\frac{3}{2}\right],\\ 2-v \text{ if } v\in\left[\frac{3}{2},2\right].\end{cases}$$

Now we apply the computation from Section 9.1 (Step 1.) and get that $\delta_P(S)=\frac{6}{7}$ for $P\in E_5\backslash E_4$.

Step 7. Suppose $P\in E_6$. Then $\tau(E_6)=2$ and the Zariski decomposition of the divisor $-K_S-vE_6\sim C+E_1+E_2+2E+2E_3+2E_4+2E_5+(2-v)E_6+E_7$ is:

$$P(v)=\begin{cases}-K_S-vE_6-\frac{v}{2}(2E_5+2E_4+2E_3+2E+E_1+E_2+E_6) \text{ if } v\in[0,1],\\ -K_S-vE_6-\frac{v}{2}(2E_5+2E_4+2E_3+2E+E_1+E_2+E_6)-(v-1)C \text{ if } v\in[1,2].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{2}(2E_5+2E_4+2E_3+2E+E_1+E_2+E_6) \text{ if } v\in[0,1],\\ \frac{v}{2}(2E_5+2E_4+2E_3+2E+E_1+E_2+E_6)+(v-1)C \text{ if } v\in[1,2].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{v^2}{2} \text{ if } v\in[0,1],\\ \frac{(2-v)^2}{2} \text{ if } v\in[1,2].\end{cases}\qquad P(v)\cdot E_6=\begin{cases}\frac{v}{2} \text{ if } v\in[0,1],\\ 1-\frac{v}{2} \text{ if } v\in[1,2].\end{cases}$$

Now we apply the computation from Section 9.1 (Step 1.) and get that $\delta_P(S)=1$ for $P\in E_6\backslash E_5$.

Step 8. Suppose $P\in E_7$. Then $\tau(E_7)=1$ and the Zariski decomposition of the divisor $-K_S-vE_7\sim C+E_1+E_2+2E+2E_3+2E_4+2E_5+2E_6+(1-v)E_7$ is given by:

$$P(v)=-K_S-vE_7-\frac{v}{2}(2E_6+2E_5+2E_4+2E_3+2E+E_1+E_2) \text{ if } v\in[0,1].$$

$$N(v)=\frac{v}{2}(2E_6+2E_5+2E_4+2E_3+2E+E_1+E_2) \text{ if } v\in[0,1].$$

Moreover,

$$(P(v))^2 = (1-v)(1+v) \text{ and } P(v)\cdot E_7 = v \text{ if } v \in [0,1].$$

Now we apply the computation from Section 9.1 (Step 2.) and get that $\delta_P(S) = \frac{3}{2}$ for $P \in E_7 \backslash E_6$. Thus, $\delta_{\mathscr{P}}(X) = \frac{3}{5}$. □

$\mathbb{E}_6$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{E}_6$ singularity at point $\mathscr{P}$. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X) = \frac{3}{5}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathscr{C}$ on S and E, E_1, E_2, E_3, E_4 and E_5 are the exceptional divisors with the intersection:

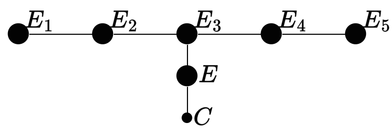


Figure 9.30: Dual graph: $(-K_S)^2 = 1$, $\mathbb{E}_6$ singularity

We have $-K_S \sim C + E_1 + 2E_2 + 3E_3 + 2E_4 + E_5 + 2E$. Let P be a point on S.

Step 1. Suppose $P \in E_3$. Then $\tau(E_3) = 3$ and the Zariski decomposition of the divisor $-K_S - vE_3 \sim C + E_1 + 2E_2 + (3-v)E_3 + 2E_4 + E_5 + 2E$ is the following:

$$P(v) = \begin{cases} -K_S - vE_3 - \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) - \frac{v}{2}E \text{ if } v \in [0,2], \\ -K_S - vE_3 - \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) - (v-1)E - (v-2)C \text{ if } v \in [2,3]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) + \frac{v}{2}E \text{ if } v \in [0,2], \\ \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) + (v-1)E + (v-2)C \text{ if } v \in [2,3]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{6} \text{ if } v \in [0,2], \\ \frac{(3-v)^2}{3} \text{ if } v \in [2,3]. \end{cases} \quad P(v)\cdot E_3 = \begin{cases} \frac{v}{6} \text{ if } v \in [0,2], \\ 1 - \frac{v}{6} \text{ if } v \in [2,3]. \end{cases}$$

Now we apply the computation from Section 9.1 (Step 1.) and get that $\delta_P(S) = \frac{3}{5}$ for $P \in E_3$.

Step 2. Suppose $P \in E_2 \cup E_4$. Without loss of generality we can assume that $P \in E_2$ since the proof is similar in other cases. Then $\tau(E_2) = 2$ and the Zariski decomposition of the divisor $-K_S - vE_2 \sim C + E_1 + (2-v)E_2 + 3E_3 + 2E_4 + E_5 + 2$ is:

$$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{2}E_1 - \frac{v}{5}(3E + 6E_3 + 4E_4 + 2E_5) \text{ if } v \in \left[0, \frac{5}{3}\right], \\ -K_S - vE_2 - \frac{v}{2}E_1 - (v-1)(3E_3 + 2E_4 + E_5) - (3v-4)E - (3v-5)C \text{ if } v \in \left[\frac{5}{3}, 2\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}E_1 + \frac{v}{5}(3E + 6E_3 + 4E_4 + 2E_5) \text{ if } v \in \left[0, \frac{5}{3}\right], \\ \frac{v}{2}E_1 + (v-1)(3E_3 + 2E_4 + E_5) + (3v-4)E + (3v-5)C \text{ if } v \in \left[\frac{5}{3}, 2\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{3v^2}{10} \text{ if } v \in \left[0, \frac{5}{3}\right], \\ \frac{3(2-v)^2}{2} \text{ if } v \in \left[\frac{5}{3}, 2\right]. \end{cases} \quad P(v) \cdot E_2 = \begin{cases} \frac{3v}{10} \text{ if } v \in \left[0, \frac{5}{3}\right], \\ 3(1 - \frac{v}{2}) \text{ if } v \in \left[\frac{5}{3}, 2\right]. \end{cases}$$

We have $S_S(E_2) = \frac{11}{9}$. Thus, $\delta_P(S) \leq \frac{9}{11}$ for $P \in E_2$. Moreover, if $P \in E_2 \backslash E_3$ we have

$$h(v) \leq \begin{cases} \frac{39v^2}{200} \text{ if } v \in \left[0, \frac{5}{3}\right], \\ \frac{3(v-2)(v-6)}{8} \text{ if } v \in \left[\frac{5}{3}, 2\right]. \end{cases}$$

Thus, $S(W^{E_2}_{\bullet,\bullet}; P) \leq \frac{7}{9} < \frac{11}{9}$. We get $\delta_P(S) = \frac{9}{11}$ for $P \in (E_2 \cup E_4) \backslash E_3$.

Step 3. Suppose $P \in E_1 \cup E_5$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:

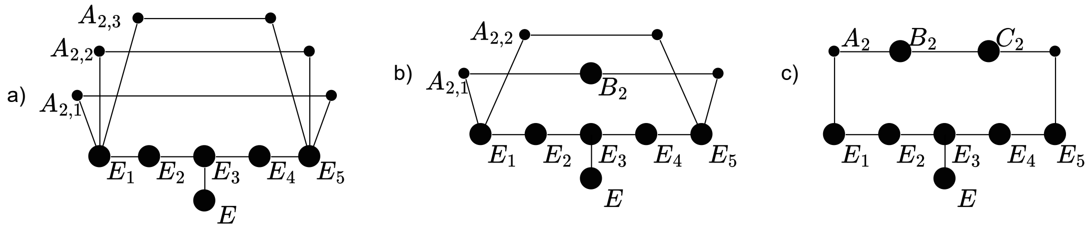


Figure 9.31: Dual graph: $(-K_S)^2 = 1$, $\mathbb{E}_6$ singularity, $\delta_P(S) = \frac{3}{5}$

Then the corresponding Zariski Decomposition of the divisor $-K_S - vE_1$ is:

a). $$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{4}(5E_2 + 6E_3 + 4E_4 + 2E_5 + 3E) \text{ if } v \in [0, 1], \\ -K_S - vE_1 - \frac{v}{4}(5E_2 + 6E_3 + 4E_4 + 2E_5 + 3E) - (v-1)(A_{2,1} + A_{2,2} + A_{2,3}) \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(5E_2 + 6E_3 + 4E_4 + 2E_5 + 3E) \text{ if } v \in [0, 1], \\ \frac{v}{4}(5E_2 + 6E_3 + 4E_4 + 2E_5 + 3E) + (v-1)(A_{2,1} + A_{2,2} + A_{2,3}) \text{ if } v \in \left[1, \frac{4}{3}\right]. \end{cases}$$

b). $P(v)=\begin{cases}-K_S-vE_1-\frac{v}{4}(5E_2+6E_3+4E_4+2E_5+3E) \text{ if } v\in[0,1],\\ -K_S-vE_1-\frac{v}{4}(5E_2+6E_3+4E_4+2E_5+3E)-(v-1)(2A_{2,1}+B_{2,1}+A_{2,2}) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$

$$N(v)=\begin{cases}\frac{v}{4}(5E_2+6E_3+4E_4+2E_5+3E) \text{ if } v\in[0,1],\\ \frac{v}{4}(5E_2+6E_3+4E_4+2E_5+3E)+(v-1)(2A_{2,1}+B_{2,1}+A_{2,2}) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$

c). $P(v)=\begin{cases}-K_S-vE_1-\frac{v}{4}(5E_2+6E_3+4E_4+2E_5+3E) \text{ if } v\in[0,1],\\ -K_S-vE_1-\frac{v}{4}(5E_2+6E_3+4E_4+2E_5+3E)-(v-1)(3A_2+B_2+C_2) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$

$$N(v)=\begin{cases}\frac{v}{4}(5E_2+6E_3+4E_4+2E_5+3E) \text{ if } v\in[0,1],\\ \frac{v}{4}(5E_2+6E_3+4E_4+2E_5+3E)+(v-1)(3A_2+B_2+C_2) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$

Then $\tau(E_1)=\frac{4}{3}$ and the Zariski Decomposition in part a). follows from

$$-K_S-vE_1\sim_{\mathbb{R}}\Big(\frac{4}{3}-v\Big)E_1+\frac{1}{3}\Big(5E_2+6E_3+4E_4+2E_5+3E+A_{2,1}+A_{2,2}+A_{2,3}\Big).$$

A similar statement holds in other parts. Moreover,

$$(P(v))^2=\begin{cases}1-\frac{3v^2}{4} \text{ if } v\in[0,1],\\ \frac{(4-3v)^2}{4} \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}\qquad P(v)\cdot E_1=\begin{cases}\frac{3v}{4} \text{ if } v\in[0,1],\\ 3(1-\frac{3v}{4}) \text{ if } v\in\left[1,\frac{4}{3}\right].\end{cases}$$

We apply the computation from Section 9.1 (Step 2.) and get $\delta_P(S)=\frac{3}{5}$ if $P\in(E_1\cup E_5)\backslash(E_2\cup E_4)$.

Step 4. Suppose $P\in E$. Then $\tau(E)=2$ and the Zariski decomposition of the divisor $-K_S-vE\sim C+E_1+2E_2+3E_3+2E_4+E_5+(2-v)E$ is:

$$P(v)=\begin{cases}-K_S-vE-\frac{v}{2}(E_1+2E_2+3E_3+2E_4+E_5) \text{ if } v\in[0,1],\\ -K_S-vE-\frac{v}{2}(E_1+2E_2+3E_3+2E_4+E_5)-(v-1)C \text{ if } v\in[1,2].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{2}(E_1+2E_2+3E_3+2E_4+E_5) \text{ if } v\in[0,1],\\ \frac{v}{2}(E_1+2E_2+3E_3+2E_4+E_5)+(v-1)C \text{ if } v\in[1,2].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{v^2}{2} \text{ if } v\in[0,1],\\ \frac{(2-v)^2}{2} \text{ if } v\in[1,2].\end{cases}\qquad P(v)\cdot E=\begin{cases}\frac{v}{2} \text{ if } v\in[0,1],\\ 1-\frac{v}{2} \text{ if } v\in[1,2].\end{cases}$$

Now we apply the computation from Section 9.1 (Step 1.) and get that $\delta_P(S)=1$ for $P\in E\backslash E_3$. Thus, $\delta_{\mathscr{P}}(X)=\frac{3}{5}$. □

$\mathbb{E}_7$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{E}_7$ singularity at point $\mathscr{P}$. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X)=\frac{3}{7}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathscr{C}$ on S and E, E_1, E_2, E_3, E_4, E_5 and E_6 are the exceptional divisors with the intersection:

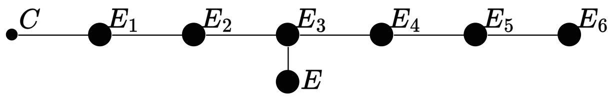


Figure 9.32: Dual graph: $(-K_S)^2=1$, $\mathbb{E}_7$ singularity

We have $-K_S\sim C+2E_1+3E_2+4E_3+3E_4+2E_5+E_6+2E$. Let P be a point on S.
Step 1. Suppose $P\in E_3$. Then $\tau(E_3)=4$ and the Zariski decomposition of the divisor $-K_S-vE_3\sim C+2E_1+3E_2+(4-v)E_3+3E_4+2E_5+E_6+2E$ is the following:

$$P(v)=\begin{cases}-K_S-vE_3-\frac{v}{4}(2E+3E_4+2E_5+E_6)-\frac{v}{3}(E_1+2E_2) \text{ if } v\in[0,3],\\ -K_S-vE_3-\frac{v}{4}(2E+3E_4+2E_5+E_6)-(v-1)E_1-(v-2)E_2-(v-3)C \text{ if } v\in[3,4].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{4}(2E+3E_4+2E_5+E_6)-\frac{v}{3}(E_1+2E_2) \text{ if } v\in[0,3],\\ \frac{v}{4}(2E+3E_4+2E_5+E_6)+(v-1)E_1+(v-2)E_2+(v-3)C \text{ if } v\in[3,4].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{v^2}{12} \text{ if } v\in[0,3],\\ \frac{(4-v)^2}{4} \text{ if } v\in[3,4].\end{cases}\quad P(v)\cdot E_3=\begin{cases}\frac{v}{12} \text{ if } v\in[0,3],\\ 1-\frac{v}{4} \text{ if } v\in[3,4].\end{cases}$$

We have $S_S(E_3)=\frac{7}{3}$. Thus, $\delta_P(S)\leq\frac{3}{7}$ for $P\in E_3$. Moreover, if $P\in E_3\cap(E\cup E_4)$ if $P\in E_3\backslash(E\cup E_4)$ we have

$$h(v)\leq\begin{cases}\frac{19v^2}{228} \text{ if } v\in\big[0,3\big],\\ \frac{(4-v)(5v+4)}{32} \text{ if } v\in[3,4]\end{cases}\quad\text{or } h(v)\leq\begin{cases}\frac{17v^2}{228} \text{ if } v\in\big[0,3\big],\\ \frac{(4-v)(7v-4)}{32} \text{ if } v\in[3,4]\end{cases}$$

Thus, $S(W^{E_3}_{\bullet,\bullet};P)\leq\frac{11}{6}<\frac{7}{3}$ or $S(W^{E_3}_{\bullet,\bullet};P)\leq\frac{5}{3}<\frac{7}{3}$. We get $\delta_P(S)=\frac{3}{7}$ for $P\in E_3$.
Step 2. Suppose $P\in E_2$. Then $\tau(E_2)=3$ and the Zariski decomposition of the divisor $-K_S-vE_2\sim C+2E_1+(3-v)E_2+4E_3+3E_4+2E_5+E_6+2E$ is the following:

$$P(v)=\begin{cases}-K_S-vE_2-\frac{v}{3}(2E+4E_3+3E_4+2E_5+E_6)-\frac{v}{2}E_1 \text{ if } v\in[0,2],\\ -K_S-vE_2-\frac{v}{3}(2E+4E_3+3E_4+2E_5+E_6)-(v-1)E_1-(v-2)C \text{ if } v\in[2,3].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{3}(2E+4E_3+3E_4+2E_5+E_6)+\frac{v}{2}E_1 \text{ if } v\in[0,2],\\ \frac{v}{3}(2E+4E_3+3E_4+2E_5+E_6)+(v-1)E_1+(v-2)C \text{ if } v\in[2,3].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{v^2}{6} \text{ if } v\in[0,2],\\ \frac{(3-v)^2}{3} \text{ if } v\in[2,3].\end{cases}\quad P(v)\cdot E_2=\begin{cases}\frac{v}{6} \text{ if } v\in[0,2],\\ 1-\frac{v}{6} \text{ if } v\in[2,3].\end{cases}$$

Now we apply the computation from Section 9.1 (Step 1.) and get that $\delta_P(S)=\frac{3}{5}$ for $P\in E_2\backslash E_3$.

Step 3. Suppose $P\in E_1$. Then $\tau(E_1)=2$ and the Zariski decomposition of the divisor $-K_S-vE_1\sim C+(2-v)E_1+3E_2+4E_3+3E_4+2E_5+E_6+2E$ is:

$$P(v)=\begin{cases}-K_S-vE_1-\frac{v}{2}(2E+3E_2+4E_3+3E_4+2E_5+E_6) \text{ if } v\in[0,1],\\ -K_S-vE_1-\frac{v}{2}(2E+3E_2+4E_3+3E_4+2E_5+E_6)-(v-1)C \text{ if } v\in[1,2].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{2}(2E+3E_2+4E_3+3E_4+2E_5+E_6) \text{ if } v\in[0,1],\\ \frac{v}{2}(2E+3E_2+4E_3+3E_4+2E_5+E_6)+(v-1)C \text{ if } v\in[1,2].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{v^2}{2} \text{ if } v\in[0,1],\\ \frac{(2-v)^2}{2} \text{ if } v\in[1,2].\end{cases}\quad P(v)\cdot E_1=\begin{cases}\frac{v}{2} \text{ if } v\in[0,1],\\ 1-\frac{v}{2} \text{ if } v\in[1,2].\end{cases}$$

Now we apply the computation from Section 9.1 (Step 1.) and get that $\delta_P(S)=1$ for $P\in E_1\backslash E_2$.

Step 4. Suppose $P\in E$. Then $\tau(E)=2$ and the Zariski decomposition of the divisor $-K_S-vE\sim C+2E_1+3E_2+4E_3+3E_4+2E_5+E_6+(2-v)E$ is:

$$P(v)=\begin{cases}-K_S-vE-\frac{v}{7}(4E_1+8E_2+12E_3+9E_4+6E_5+3E_6) \text{ if } v\in\left[0,\frac{7}{4}\right],\\ -K_S-vE-(4v-7)C-(4v-6)E_1-(4v-5)E_2-(v-1)(4E_3+3E_4+2E_5+E_6) \text{ if } v\in\left[\frac{7}{4},2\right].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{7}(4E_1+8E_2+12E_3+9E_4+6E_5+3E_6) \text{ if } v\in\left[0,\frac{7}{4}\right],\\ (4v-7)C+(4v-6)E_1+(4v-5)E_2+(v-1)(4E_3+3E_4+2E_5+E_6) \text{ if } v\in\left[\frac{7}{4},2\right].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{2v^2}{7} \text{ if } v\in\left[0,\frac{7}{4}\right],\\ 2(2-v)^2 \text{ if } v\in\left[\frac{7}{4},2\right].\end{cases}\quad P(v)\cdot E=\begin{cases}\frac{2v}{7} \text{ if } v\in\left[0,\frac{7}{4}\right],\\ 2(2-v) \text{ if } v\in\left[\frac{7}{4},2\right].\end{cases}$$

We have $S_S(E) = \frac{5}{4}$. Thus, $\delta_P(S) \le \frac{4}{5}$ for $P \in E$. Moreover, if $P \in E \backslash E_3$ we have

$$h(v) \le \begin{cases} \frac{2v^2}{49} \text{ if } v \in \left[0, \frac{7}{4}\right], \\ 2(v-2)^2 \text{ if } v \in \left[\frac{7}{4}, 2\right]. \end{cases}$$

Thus $S(W^{E_2}_{\bullet,\bullet}; P) \le \frac{1}{6} < \frac{5}{4}$. We get $\delta_P(S) = \frac{4}{5}$ for $P \in E \backslash E_3$.

Step 5. Suppose $P \in E_4$. Then $\tau(E_4) = 3$ and the Zariski decomposition of the divisor $-K_S - vE_4 \sim C + 2E_1 + 3E_2 + 4E_3 + (3-v)E_4 + 2E_5 + E_6 + 2E$ is:

$$P(v) = \begin{cases} -K_S - vE_4 - \frac{v}{5}(2E_1 + 4E_2 + 6E_3 + 3E) - \frac{v}{3}(2E_5 + E_6) \text{ if } v \in \left[0, \frac{5}{2}\right], \\ -K_S - vE_4 - (2v-5)C - (2v-4)E_1 - (2v-3)E_2 - (2v-2)E_3 - (v-1)E - \frac{v}{3}(2E_5 + E_6) \text{ if } v \in \left[\frac{5}{2}, 3\right]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(2E_1 + 4E_2 + 6E_3 + 3E) - \frac{v}{3}(2E_5 + E_6) \text{ if } v \in \left[0, \frac{5}{2}\right], \\ (2v-5)C + (2v-4)E_1 + (2v-3)E_2 + (2v-2)E_3 + (v-1)E + \frac{v}{3}(2E_5 + E_6) \text{ if } v \in \left[\frac{5}{2}, 3\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{2v^2}{15} \text{ if } v \in \left[0, \frac{5}{2}\right], \\ \frac{2(3-v)^2}{3} \text{ if } v \in \left[\frac{5}{2}, 3\right]. \end{cases} \quad P(v) \cdot E_4 = \begin{cases} \frac{2v}{15} \text{ if } v \in \left[0, \frac{5}{2}\right], \\ 2(1 - \frac{v}{3}) \text{ if } v \in \left[\frac{5}{2}, 3\right]. \end{cases}$$

We have $S_S(E_4) = \frac{11}{6}$. Thus, $\delta_P(S) \le \frac{6}{11}$ for $P \in E_4$. Moreover, if $P \in E_4 \backslash E_3$ we have

$$h(v) \le \begin{cases} \frac{22v^2}{225} \text{ if } v \in \left[0, \frac{5}{2}\right], \\ \frac{2(3-v)(v+3)}{9} \text{ if } v \in \left[\frac{5}{2}, 3\right]. \end{cases}$$

Thus, $S(W^{E_4}_{\bullet,\bullet}; P) \le \frac{4}{3} < \frac{11}{6}$. We get $\delta_P(S) = \frac{6}{11}$ for $P \in E_4 \backslash E_3$.

Step 6. Suppose $P \in E_5$. Then $\tau(E_5) = 2$ and the Zariski decomposition of the divisor $-K_S - vE_5 \sim C + 2E_1 + 3E_2 + 4E_3 + 3E_4 + (2-v)E_5 + E_6 + 2E$ is given by:

$$P(v) = -K_S - vE_5 - \frac{v}{4}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 3E + 2E_6) \text{ if } v \in [0, 2].$$
$$N(v) = \frac{v}{4}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 3E + 2E_6) \text{ if } v \in [0, 2].$$

Moreover,

$$(P(v))^2 = \frac{(2-v)(2+v)}{4} \text{ and } P(v) \cdot E_5 = \frac{v}{4} \text{ if } v \in [0, 2].$$

Now we apply the computation from Section 9.1 (Step 1.) and get that $\delta_P(S) = \frac{3}{4}$ for $P \in E_5 \backslash E_4$.

Step 7. Suppose $P \in E_6$. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:

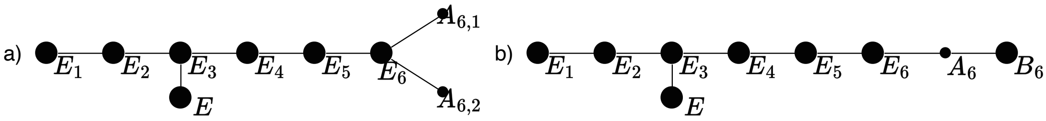


Figure 9.33: Dual graph: $(-K_S)^2 = 1$, $\mathbb{E}_7$ singularity, $\delta_P(S) = \frac{6}{5}$

Then $\tau(E_6) = \frac{3}{2}$ and the Zariski Decomposition of the divisor $-K_S - vE_6$ is:

a).
$$P(v) = \begin{cases} -K_S - vE_6 - \frac{v}{3}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E) \text{ if } v \in [0,1], \\ -K_S - vE_6 - \frac{v}{3}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E) - (v-1)(A_{6,1} + A_{6,2}) \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{3}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E) \text{ if } v \in [0,1], \\ \frac{v}{3}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E) + (v-1)(A_{1,1} + A_{1,2}) \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vE_6 - \frac{v}{3}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E) \text{ if } v \in [0,1], \\ -K_S - vE_6 - \frac{v}{3}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E) - (v-1)(2A_6 + B_6) \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{3}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E) \text{ if } v \in [0,1], \\ \frac{v}{3}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E) + (v-1)(2A_6 + B_6) \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

The Zariski Decomposition in part a). follows from

$$-K_S - vE_6 \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)E_6 + \frac{1}{2}\Big(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E + A_{6,1} + A_{6,2}\Big).$$

A similar statement holds in other parts. Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{2v^2}{3} \text{ if } v \in [0,1], \\ \frac{(3-2v)^2}{3} \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases} \quad P(v) \cdot E_1 = \begin{cases} \frac{2v}{3} \text{ if } v \in [0,1], \\ 2(1 - \frac{2v}{3}) \text{ if } v \in \left[1, \frac{3}{2}\right]. \end{cases}$$

Now we apply the computation from Section 9.1 (Step 2.) and get that $\delta_P(S) = \frac{6}{5}$ for $P \in E_6 \backslash E_5$. Thus, $\delta_{\mathscr{P}}(X) = \frac{3}{7}$. □

$\mathbb{E}_8$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{E}_8$ singularity at point $\mathscr{P}$. Let $\mathscr{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathscr{P}$. Then $\delta_{\mathscr{P}}(X) = \frac{3}{11}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathscr{C}$ on S and E, E_1, E_2, E_3, E_4, E_5, E_6 and E_7 are the exceptional divisors with the intersection:

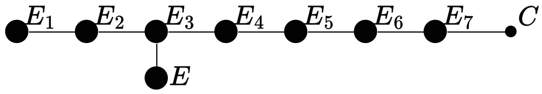


Figure 9.34: Dual graph: $(-K_S)^2=1$, $\mathbb{E}_8$ singularity

We have $-K_S \sim C+2E_1+4E_2+6E_3+5E_4+4E_5+3E_6+2E_7+3E$.

Step 1. Suppose $P\in E_3$. Then $\tau(E_3)=6$ and the Zariski decomposition of the divisor $-K_S-vE_3\sim C+2E_1+4E_2+(6-v)E_3+5E_4+4E_5+3E_6+2E_7+3E$ is the following:

$$P(v)=\begin{cases}-K_S-vE_3-\frac{v}{2}E-\frac{v}{3}(E_1+2E_2)-\frac{v}{5}(4E_4+3E_5+2E_6+E_7) \text{ if } v\in\big[0,5\big],\\ -K_S-vE_3-\frac{v}{2}E-\frac{v}{3}(E_1+2E_2)-(v-1)E_4-(v-2)E_5-(v-3)E_6-(v-4)E_7-(v-5)C \text{ if } v\in[5,6].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{2}E+\frac{v}{3}(E_1+2E_2)+\frac{v}{5}(4E_4+3E_5+2E_6+E_7) \text{ if } v\in\big[0,5\big],\\ \frac{v}{2}E+\frac{v}{3}(E_1+2E_2)+(v-1)E_4+(v-2)E_5+(v-3)E_6+(v-4)E_7+(v-5)C \text{ if } v\in[5,6].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{v^2}{30} \text{ if } v\in[0,5],\\ \frac{(6-v)^2}{6} \text{ if } v\in[5,6].\end{cases}\quad P(v)\cdot E_3=\begin{cases}\frac{v}{30} \text{ if } v\in[0,5],\\ 1-\frac{v}{6} \text{ if } v\in[5,6].\end{cases}$$

We have $S_S(E_3)=\frac{11}{3}$. Thus, $\delta_P(S)\le\frac{3}{11}$ for $P\in E_3$. Moreover, if $P\in E_3\cap(E\cup E_2)$ if $P\in E_3\backslash(E\cup E_2)$ we have

$$h(v)\le\begin{cases}\frac{41v^2}{1800} \text{ if } v\in[0,5],\\ \frac{(6-v)(7v+6)}{72} \text{ if } v\in[5,6].\end{cases}\quad \text{or } h(v)\le\begin{cases}\frac{49v^2}{1800} \text{ if } v\in[0,5],\\ \frac{(6-v)(11v-6)}{72} \text{ if } v\in[5,6].\end{cases}$$

Thus, $S(W^{E_3}_{\bullet,\bullet};P)\le\frac{5}{2}<\frac{11}{3}$ or $S(W^{E_3}_{\bullet,\bullet};P)\le 3<\frac{11}{3}$. We get $\delta_P(S)=\frac{3}{11}$ for $P\in E_3$.

Step 2. Suppose $P\in E_2$. Then $\tau(E_2)=4$ and the Zariski decomposition of the divisor $-K_S-vE_2\sim C+2E_1+(4-v)E_2+6E_3+5E_4+4E_5+3E_6+2E_7+3E$ is:

$$P(v)=\begin{cases}-K_S-vE_2-\frac{v}{2}E_1-\frac{v}{7}(5E+10E_3+8E_4+6E_5+4E_6+2E_7) \text{ if } v\in\big[0,\frac{7}{2}\big],\\ -K_S-vE_2-\frac{v}{2}E_1-(v-1)E-(2v-2)E_3-(2v-3)E_4-(2v-4)E_5-(2v-5)E_6-(2v-6)E_7-(2v-7)C \text{ if } v\in\big[\frac{7}{2},4\big].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{2}E_1+\frac{v}{7}(5E+10E_3+8E_4+6E_5+4E_6+2E_7) \text{ if } v\in\big[0,\frac{7}{2}\big],\\ \frac{v}{2}E_1+(v-1)E+(2v-2)E_3+(2v-3)E_4+(2v-4)E_5+(2v-5)E_6+(2v-6)E_7+(2v-7)C \text{ if } v\in\big[\frac{7}{2},4\big].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{v^2}{14} \text{ if } v\in\big[0,\frac{7}{2}\big],\\ \frac{(4-v)^2}{2} \text{ if } v\in\big[\frac{7}{2},4\big].\end{cases}\quad P(v)\cdot E_2=\begin{cases}\frac{v}{14} \text{ if } v\in\big[0,\frac{7}{2}\big],\\ 2-\frac{v}{2} \text{ if } v\in\big[\frac{7}{2},4\big].\end{cases}$$

We have $S_S(E_2)=\frac{5}{2}$. Thus, $\delta_P(S)\le\frac{2}{5}$ for $P\in E_2$. Moreover, if $P\in E_2\backslash E_3$ we have

$$h(v)\le\begin{cases}\frac{15v^2}{392} \text{ if } v\in\left[0,\frac{7}{2}\right],\\ \frac{(4-v)(4+v)}{8} \text{ if } v\in\left[\frac{7}{2},4\right].\end{cases}$$

Thus, $S(W^{E_2}_{\bullet,\bullet};P)\le\frac{4}{3}<\frac{5}{2}$. We get $\delta_P(S)=\frac{2}{5}$ for $P\in E_2\backslash E_3$.
Step 3. Suppose $P\in E_1$. Then $\tau(E_1)=2$ and the Zariski decomposition of the divisor $-K_S-vE_1\sim C+(2-v)E_1+4E_2+6E_3+5E_4+4E_5+3E_6+2E_7+3E$ is given by:

$$P(v)=-K_S-vE_1-\frac{v}{4}(5E+7E_2+10E_3+8E_4+6E_5+4E_6+2E_7) \text{ if } v\in[0,2].$$
$$N(v)=\frac{v}{4}(5E+7E_2+10E_3+8E_4+6E_5+4E_6+2E_7) \text{ if } v\in[0,2].$$

Moreover,

$$(P(v))^2=\frac{(2-v)(2+v)}{4} \text{ and } P(v)\cdot E_1=\frac{v}{4} \text{ if } v\in[0,2].$$

Now we apply the computation from Section 9.1 (Step 1.) and get that $\delta_P(S)=\frac{3}{4}$ for $P\in E_1\backslash E_2$.
Step 4. Suppose $P\in E$. Then $\tau(E)=3$ and the Zariski decomposition of the divisor $-K_S-vE\sim C+2E_1+4E_2+6E_3+5E_4+4E_5+3E_6+2E_7+(3-v)E$ is:

$$P(v)=\begin{cases}-K_S-vE-\frac{v}{8}(5E_1+10E_2+15E_3+12E_4+9E_5+6E_6+3E_7) \text{ if } v\in\left[0,\frac{8}{3}\right],\\ -K_S-vE-(v-1)(E_1+2E_2+3E_3)-(3v-4)E_4-(3v-5)E_5-(3v-6)E_6-(3v-7)E_7-(3v-8)C \text{ if } v\in\left[\frac{8}{3},3\right].\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{8}(5E_1+10E_2+15E_3+12E_4+9E_5+6E_6+3E_7) \text{ if } v\in\left[0,\frac{8}{3}\right],\\ (v-1)(E_1+2E_2+3E_3)+(3v-4)E_4+(3v-5)E_5+(3v-6)E_6+(3v-7)E_7+(3v-8)C \text{ if } v\in\left[\frac{8}{3},3\right].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{v^2}{8} \text{ if } v\in\left[0,\frac{8}{3}\right],\\ (3-v)^2 \text{ if } v\in\left[\frac{8}{3},3\right].\end{cases}\qquad P(v)\cdot E=\begin{cases}\frac{v}{8} \text{ if } v\in\left[0,\frac{8}{3}\right],\\ 3-v \text{ if } v\in\left[\frac{8}{3},3\right].\end{cases}$$

We have $S_S(E)=\frac{17}{9}$. Thus, $\delta_P(S)\le\frac{9}{17}$ for $P\in E$. Moreover, if $P\in E\backslash E_3$ we have

$$h(v)\le\begin{cases}\frac{v^2}{128} \text{ if } v\in\left[0,\frac{8}{3}\right],\\ \frac{(3-v)^2}{2} \text{ if } v\in \text{ if } v\in\left[\frac{8}{3},3\right].\end{cases}$$

Thus, $S(W^{E}_{\bullet,\bullet};P)\le\frac{1}{9}<\frac{17}{9}$. We get $\delta_P(S)=\frac{9}{17}$ for $P\in E\backslash E_3$.
Step 5. Suppose $P\in E_4$. Then $\tau(E_4)=5$ and the Zariski decomposition of the divisor $-K_S-vE_4\sim C+2E_1+4E_2+6E_3+(5-v)E_4+4E_5+3E_6+2E_7+3E$ is the following:

$$P(v)=\begin{cases}-K_S-vE_4-\frac{v}{5}(2E_1+4E_2+6E_3+3E)-\frac{v}{4}(3E_5+2E_6+E_7) \text{ if } v\in[0,4],\\ -K_S-vE_4-\frac{v}{5}(2E_1+4E_2+6E_3+3E)-(v-1)E_5-(v-2)E_6-(v-3)E_7-(v-4)C \text{ if } v\in[4,5].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{5}(2E_1+4E_2+6E_3+3E)+\frac{v}{4}(3E_5+2E_6+E_7) \text{ if } v\in[0,4],\\ \frac{v}{5}(2E_1+4E_2+6E_3+3E)+(v-1)E_5+(v-2)E_6+(v-3)E_7+(v-4)C \text{ if } v\in[4,5].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{v^2}{20} \text{ if } v\in[0,4],\\ \frac{(5-v)^2}{5} \text{ if } v\in[4,5].\end{cases}\quad P(v)\cdot E_4=\begin{cases}\frac{v}{20} \text{ if } v\in[0,4],\\ 1-\frac{v}{5} \text{ if } v\in[4,5].\end{cases}$$

We have $S_S(E_4)=3$. Thus, $\delta_P(S)\le\frac{1}{3}$ for $P\in E_4$. Moreover, if $P\in E_4\backslash E_3$ we have

$$h(v)\le\begin{cases}\frac{31v^2}{800} \text{ if } v\in[0,4],\\ \frac{(5-v)(9v-5)}{50} \text{ if } v\in[4,5].\end{cases}$$

Thus, $S(W^{E_4}_{\bullet,\bullet};P)\le\frac{7}{3}<3$. We get $\delta_P(S)=3$ for $P\in E_4\backslash E_3$.

Step 6. Suppose $P\in E_5$. Then $\tau(E_5)=4$ and the Zariski decomposition of the divisor $-K_S-vE_5\sim C+2E_1+4E_2+6E_3+5E_4+(4-v)E_5+3E_6+2E_7+3E$ is the following:

$$P(v)=\begin{cases}-K_S-vE_5-\frac{v}{4}(2E_1+4E_2+6E_3+5E_4+3E)-\frac{v}{3}(2E_6+E_7) \text{ if } v\in[0,3],\\ -K_S-vE_5-\frac{v}{4}(2E_1+4E_2+6E_3+5E_4+3E)-(v-1)E_6-(v-2)E_7-(v-3)C \text{ if } v\in[3,4].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{4}(2E_1+4E_2+6E_3+5E_4+3E)-\frac{v}{3}(2E_6+E_7) \text{ if } v\in[0,3],\\ \frac{v}{4}(2E_1+4E_2+6E_3+5E_4+3E)+(v-1)E_6+(v-2)E_7+(v-3)C \text{ if } v\in[3,4].\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{v^2}{12} \text{ if } v\in[0,3],\\ \frac{(4-v)^2}{4} \text{ if } v\in[3,4].\end{cases}\quad P(v)\cdot E_5=\begin{cases}\frac{v}{12} \text{ if } v\in[0,3],\\ 1-\frac{v}{4} \text{ if } v\in[3,4].\end{cases}$$

Now we apply the computation from Section 9.1 (Step 1.) and get that $\delta_P(S)=\frac{3}{7}$ for $P\in E_5\backslash E_4$.

Step 7. Suppose $P\in E_6$. Then $\tau(E_6)=3$ and the Zariski decomposition of the divisor $-K_S-vE_6\sim C+2E_1+4E_2+6E_3+5E_4+4E_5+(3-v)E_6+2E_7+3E$ is the following:

$$P(v)=\begin{cases}-K_S-vE_6-\frac{v}{3}(2E_1+4E_2+6E_3+5E_4+4E_5+3E)-\frac{v}{2}E_7 \text{ if } v\in[0,2],\\ -K_S-vE_6-\frac{v}{3}(2E_1+4E_2+6E_3+5E_4+4E_5+3E)-(v-1)E_7-(v-2)C \text{ if } v\in[2,3].\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{3}(2E_1+4E_2+6E_3+5E_4+4E_5+3E)+\frac{v}{2}E_7 \text{ if } v\in[0,2],\\ \frac{v}{3}(2E_1+4E_2+6E_3+5E_4+4E_5+3E)+(v-1)E_7+(v-2)C \text{ if } v\in[2,3].\end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{6} \text{ if } v \in [0,2], \\ \frac{(3-v)^2}{3} \text{ if } v \in [2,3]. \end{cases} \quad P(v) \cdot E_6 = \begin{cases} \frac{v}{6} \text{ if } v \in [0,2], \\ 1 - \frac{v}{6} \text{ if } v \in [2,3]. \end{cases}$$

Now we apply the computation from Section 9.1 (Step 1.) and get that $\delta_P(S) = \frac{3}{5}$ for $P \in E_6 \backslash E_5$.

Step 8. Suppose $P \in E_7$. Then $\tau(E_7) = 2$ and the Zariski decomposition of the divisor $-K_S - vE_7 \sim C + 2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + (2-v)E_7 + 3E$ is:

$$P(v) = \begin{cases} -K_S - vE_7 - \frac{v}{2}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 3E) \text{ if } v \in [0,1], \\ -K_S - vE_7 - \frac{v}{2}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 3E) - (v-1)C \text{ if } v \in [1,2]. \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 3E) \text{ if } v \in [0,1], \\ \frac{v}{2}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 3E) + (v-1)C \text{ if } v \in [1,2]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{2} \text{ if } v \in [0,1], \\ \frac{(2-v)^2}{2} \text{ if } v \in [1,2]. \end{cases} \quad P(v) \cdot E_7 = \begin{cases} \frac{v}{2} \text{ if } v \in [0,1], \\ 1 - \frac{v}{2} \text{ if } v \in [1,2]. \end{cases}$$

Now we apply the computation from Section 9.1 (Step 1.) and get that $\delta_P(S) = 1$ for $P \in E_7 \backslash E_6$.

Thus, $\delta_{\mathscr{P}}(X) = \frac{3}{11}$. □

PART II

K-stability of Fano Threefolds

Chapter 10

First Examples of K-stable Fano Threefolds

In this chapter, we discuss the immediate applications of the results described in Part I.

10.1 K-stable Fano Surfaces

Let X be a del Pezzo surface of degree 2 with at most Du Val singularities. Let S be a weak resolution of X. We will call an image on X of a (-1)-curve in S **a line** as was done in Cheltsov and Prokhorov (2021). The immediate corollaries from Main Theorem of Part I are:

Corollary 10.1.1. *Let X be a Du Val del Pezzo surface of degree 4 with at most $\mathbb{A}_1$ singularities then X is K-semi-stable.*

Proof. For such X have $\delta(X) \geq 1$. Thus, X is K-semi-stable by (Araujo et al., 2023, Theorem 1.59). □

Corollary 10.1.2. *Let X be a Du Val del Pezzo surface of degree 3 with at most $\mathbb{A}_2$ singularities then X is K-semi-stable.*

Proof. For such X have $\delta(X) \geq 1$. Thus, X is K-semi-stable by (Araujo et al., 2023, Theorem 1.59). □

Corollary 10.1.3 (Odaka et al. (2016), Cheltsov and Prokhorov (2021))**.** *Let X be a Du Val del Pezzo surface of degree 3 with at most $\mathbb{A}_1$ singularities then X is K-stable. Moreover, $\mathrm{Aut}(X)$ is finite.*

Proof. For such X have $\delta(X) > 1$. Thus, X is K-stable. X is K-stable so by (Blum & Xu, 2019, Corollary 1.3) $\mathrm{Aut}(X)$ is finite. □

Corollary 10.1.4. *Let X be a Du Val del Pezzo surface of degree 2 with at most $\mathbb{A}_3$ singularities then X is K-semi-stable.*

Proof. For such X have $\delta(X) \geq 1$. Thus, X is K-semi-stable by (Araujo et al., 2023, Theorem 1.59). □

Corollary 10.1.5 (Ghigi and Kollár (2007); Odaka et al. (2016), Cheltsov and Prokhorov (2021)). *Let X be a Du Val del Pezzo surface of degree 2 with at most $\mathbb{A}_2$ singularities then X is K-stable. Moreover, $\mathrm{Aut}(X)$ is finite.*

Proof. For such X have $\delta(X) > 1$. Thus, X is K-stable. This was studied in Ghigi and Kollár (2007). X is K-stable so by (Blum & Xu, 2019, Corollary 1.3) $\mathrm{Aut}(X)$ is finite □

Corollary 10.1.6 (Odaka et al. (2016)). *Let X be a Du Val del Pezzo surface of degree 2 with at least one $\mathbb{A}_3$ singularity. Then X is K-polystable if it has two $\mathbb{A}_3$ singularities and X is strictly K-semi-stable otherwise.*

Proof. Suppose X has at least one $\mathbb{A}_3$ singularity. By Cheltsov and Prokhorov (2021) only surfaces of types $2\mathbb{A}_3$ and $2\mathbb{A}_3 + \mathbb{A}_1$ have infinite automorphism groups. Suppose X has exactly two $\mathbb{A}_3$ singularities, $G = \mathrm{Aut}(X)$. Then there are no G-invariant points that belong to lines on X and for all G-invariant points we have $\delta_P(X) > 1$ so by (Zhuang, 2021, Corollary 4.14) X is K-polystable. Any X with exactly one $\mathbb{A}_3$-singularity deforms isotrivially to this case. Thus, if X has exactly one $\mathbb{A}_3$-singularity it is strictly K-semi-stable by (Araujo et al., 2023, Corollary 1.13). Suppose now that X is of type $2\mathbb{A}_3\mathbb{A}_1$ and $G = \mathrm{Aut}(X)$ then the only G-invariant points that belong to lines on X is a unique $\mathbb{A}_1$ point so for all G-invariant points we have $\delta_P(X) > 1$ and by (Zhuang, 2021, Corollary 4.14) X is K-polystable. □

Corollary 10.1.7. *Let X be a Du Val del Pezzo surface of degree 1 with $\mathbb{A}_n$ ($n \leq 8$) or $\mathbb{D}_4$ singularities then X is K-semi-stable.*

Proof. For such X have $\delta(X) \geq 1$. Thus, X is K-semi-stable by (Araujo et al., 2023, Theorem 1.59). □

Corollary 10.1.8 (Odaka et al. (2016), Cheltsov and Prokhorov (2021)). *Let X be a Du Val del Pezzo surface of degree 1 with at most $\mathbb{A}_6$ singularities or a Du Val del Pezzo surface of degree 1 with $\mathbb{A}_7$ singularity and irreducible ramification divisor R then X is K-stable. Moreover, $\mathrm{Aut}(X)$ is finite.*

Proof. For such X have $\delta(X) > 1$. Thus, X is K-stable. By (Blum & Xu, 2019, Corollary 1.3) $\mathrm{Aut}(X)$ is finite for K-stable X. □

10.2 First examples of K-stable Fano Threefolds

There are also applications in the case of Fano threefolds. Smooth Fano threefolds over $\mathbb{C}$ were classified in Iskovskikh (1997, 1998) and Mori and Mukai (1981, 2003) into 105 families. The detailed description of these families can be found in Araujo et al. (2023) where the problem to find all K-polystable smooth Fano threefolds in each family was posed. The output of this paper, give some alternative proofs for this problem as well as some proofs in case of singular

Fano threefolds. We know (Fujita (2019); Li (2017)) that the Fano threefold $\mathbf{X}$ is K-stable if and only if for every prime divisor $\mathbf{E}$ over $\mathbf{X}$ we have

$$\beta(\mathbf{E}) = A_{\mathbf{X}}(\mathbf{E}) - S_{\mathbf{X}}(\mathbf{E}) > 0,$$

where $A_{\mathbf{X}}(\mathbf{E})$ is the log discrepancy of the divisor $\mathbf{E}$ and $S_{\mathbf{X}}\big(\mathbf{E}\big) = \frac{1}{(-K_{\mathbf{X}})^3}\int\limits_0^\infty \mathrm{vol}\big(-K_{\mathbf{X}} - u\mathbf{E}\big)du.$ To show this, we fix a prime divisor $\mathbf{E}$ over $\mathbf{X}$. Then we set $Z = C_{\mathbf{X}}(\mathbf{E})$. Let Q be a general point in Z. Following Abban and Zhuang (2022); Araujo et al. (2023) denote

$$\delta_Q\big(X, W^X_{\bullet,\bullet}\big) = \inf_{\substack{F/X \\ Q\in C_X(F)}} \frac{A_X(F)}{S\big(W^X_{\bullet,\bullet};F\big)} \text{ and } \delta_Q\big(\mathbf{X}\big) = \inf_{\substack{\mathbf{F}/\mathbf{X} \\ Q\in C_{\mathbf{X}}(\mathbf{F})}} \frac{A_{\mathbf{X}}(\mathbf{F})}{S_{\mathbf{X}}(\mathbf{F})},$$

where the first infimum is taken by all prime divisors F over the surface X whose center on X contains Q and the second infimum is taken by all prime divisors $\mathbf{F}$ over the threefold $\mathbf{X}$ whose center on $\mathbf{X}$ contains Q. Moreover, it follows from (Araujo et al., 2023, Theorem 1.95) that

$$\delta_Q(\mathbf{X}) \geqslant \min\Big\{\frac{1}{S_{\mathbf{X}}(X)}, \delta_Q\big(X; W^X_{\bullet,\bullet}\big)\Big\}.$$

10.2.1 Family №1.13 (Del Pezzo Threefolds of degree 3)

Let $\mathbf{V}$ be a Fano threefold with canonical Gorenstein singularities such that $-K_{\mathbf{V}} \sim 2H$ for some $H \in \mathrm{Pic}(\mathbf{V})$ with $H^3 = 3$. Then $\mathbf{V}$ is a cubic hypersurface in $\mathbb{P}^4$ and a del Pezzo threefold of degree 3. A general element in $|H|$ is a Du Val del Pezzo surface of degree 3 and if $\mathbf{V}$ has isolated singularities then a general surface in $|H|$ is a smooth.

Remark 10.2.1. Every smooth element in Family №1.13. is known to be K-stable by Araujo et al. (2023).

Main Theorem gives the following corollary:

COROLLARY 1. Suppose that for any point Q on $\mathbf{V}$ there exists an element $X \in |H|$ such that $Q \in X$ and X is smooth then $\mathbf{V}$ is K-stable.

Proof. Suppose X is an irreducible element of $|H|$ then $S_{\mathbf{V}}(X) < 1$. As explained above we fix a prime divisor $\mathbf{E}$ over $\mathbf{V}$. Then we set $Z = C_{\mathbf{V}}(\mathbf{E})$ and if $\beta(\mathbf{E}) \leqslant 0$, then $\delta_Q(X, W^X_{\bullet,\bullet}) \leqslant 1$. Let Q be a general point in Z, Let X be the general element of $|H|$ that contains Q. The divisor $-K_{\mathbf{V}} - uX$ is nef if and only if $u \leq 2$ and the Zariski Decomposition is given by by $P(u) = -K_{\mathbf{V}} - uX \sim (2-u)X$ and $N(u) = 0$ for $u \in [0,2]$. By (Araujo et al., 2023, Corollary 1.110) for any divisor F such that $O \in C_X(F)$ over X we get:

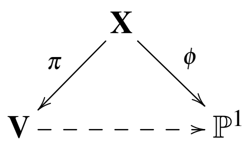
$$S\big(W^X_{\bullet,\bullet};F\big) =$$

$$= \frac{3}{(-K_{\mathbf{V}})^3}\left(\int_0^{\tau}\big(P(u)^2\cdot X\big)\cdot \mathrm{ord}_Q\Big(N(u)\big|_X\Big)du + \int_0^{\tau}\int_0^{\infty}\mathrm{vol}\big(P(u)\big|_X - vF\big)dvdu\right) =$$
$$= \frac{3}{24}\int_0^{\tau}\int_0^{\infty}\mathrm{vol}\big(P(u)\big|_X - vF\big)dvdu = \frac{3}{24}\int_0^{2}(2-u)^3\int_0^{\infty}\mathrm{vol}\big(-K_X - wF\big)dwdu =$$
$$= \frac{3}{8}\int_0^{2}(2-u)^3\Big(\frac{1}{3}\int_0^{\infty}\mathrm{vol}\big(-K_X - wF\big)dw\Big)du = \frac{3}{8}\int_0^{2}(2-u)^3 S_X(F)du =$$
$$= \frac{3}{2}S_X(F) \leq \frac{3}{2}\frac{A_X(F)}{\delta_Q(X)}.$$

We get that $\delta_Q(\mathbf{V}) \geq \frac{2}{3}\delta_Q(X)$. For smooth X we have $\delta_Q(X) \geq \frac{3}{2}$. If Q is a singular point and there exists an element X of $|H|$ with $\delta_Q(X) = \frac{3}{2}$ then $\frac{A_{\mathbf{X}}(\mathbf{E})}{S_{\mathbf{X}}(\mathbf{E})} > \min\Big\{\frac{1}{S_{\mathbf{X}}(X)}, \delta_Q\big(X, W^X_{\bullet,\bullet}\big)\Big\}$ from (Araujo et al., 2023, Corollary 1.108.) and otherwise we choose X with $\delta_Q(X) > \frac{3}{2}$ so $\delta_Q(\mathbf{V}) > 1$ if X is smooth and the result follows. □

10.2.2 Family №2.5

Let $\mathbf{V}$ be a Fano threefold with canonical Gorenstein singularities such that $-K_{\mathbf{V}} \sim 2H$ for some $H \in \mathrm{Pic}(\mathbf{V})$ with $H^3 = 3$. Then $\mathbf{V}$ is a cubic hypersurface in $\mathbb{P}^4$ and a del Pezzo threefold of degree 3. Let S_1 and S_2 be two distinct surfaces in the linear system $|H|$, and let $\mathscr{C} = S_1 \cap S_2$. Suppose that the curve $\mathscr{C}$ is smooth. Then $\mathscr{C}$ is an elliptic curve by the adjunction formula. Let $\pi : \mathbf{X} \to \mathbf{V}$ be the blow up of the curve $\mathscr{C}$, and let E be the π-exceptional surface. We have the following commutative diagram:



Where $\mathbf{V} \dashrightarrow \mathbb{P}^1$ is the rational map given by the pencil that is generated by S_1 and S_2, and ϕ is a fibration into del Pezzo surfaces of degree 3.

Remark 10.2.2. Every smooth threefold in Family 2.5 such that there is no fiber of p_1 which contains $\mathbb{D}_5$ or $\mathbb{E}_6$ singularity in this family is known to be K-stable Cheltsov et al. (2024).

Main Theorem gives the following corollary:

COROLLARY 2. If every fiber X of ϕ at most $\mathbb{A}_2$ singularities, then $\mathbf{X}$ is K-stable.

Proof. If X is an irreducible fiber of p_1 then we have $S_{\mathbf{X}}(X) < 1$. We now fix a prime divisor $\mathbf{E}$ over $\mathbf{X}$. Then we set $Z = C_{\mathbf{X}}(\mathbf{E})$. Let Q be the point on Z. let X be the fiber of ϕ that passes through Q. Then $-K_{\mathbf{X}} - uX$ is nef if and only if $u \leq 2$ and the Zariski Decomposition is given by

$$P(u) = \begin{cases} -K_{\mathbf{X}} - uX \sim (2-u)X + E \text{ if } u \in [0,1], \\ -K_{\mathbf{X}} - uX - (u-1)E \sim (2-u)\pi^*(H) \text{ if } u \in [1,2]. \end{cases}$$

$$N(u)=\begin{cases}0 \text{ if } u\in[0,1],\\ (u-1)E \text{ if } u\in[1,2].\end{cases}$$

We apply Abban-Zhuang method to prove that $Q\not\in E\cong\mathscr{C}\times\mathbb{P}^1$. By (Araujo et al., 2023, Corollary 1.110) for any divisor F such that $Q\in C_X(F)$ over X we get:

$$\begin{aligned}
&S\big(W^X_{\bullet,\bullet};F\big)=\\
&=\frac{3}{(-K_{\mathbf{X}})^3}\left(\int_0^\tau\big(P(u)^2\cdot X\big)\cdot\mathrm{ord}_Q\Big(N(u)\big|_X\Big)du+\int_0^\tau\int_0^\infty\mathrm{vol}\big(P(u)\big|_X-vF\big)dvdu\right)=\\
&=\frac{3}{12}\int_0^\tau\int_0^\infty\mathrm{vol}\big(P(u)\big|_X-vF\big)dvdu=\\
&=\frac{3}{12}\left(\int_0^1\int_0^\infty\mathrm{vol}\big(-K_X-vF\big)dvdu+\int_1^2\int_0^\infty\mathrm{vol}\big(-K_X-(u-1)E|_X-vF\big)dvdu\right)=\\
&=\frac{3}{12}\left(\int_0^\infty\mathrm{vol}\big(-K_X-vF\big)dv+\int_1^2(2-u)^3\int_0^\infty\mathrm{vol}\big(-K_X-wF\big)dwdu\right)=\\
&=\frac{3}{4}\left(\frac{1}{3}\int_0^\infty\mathrm{vol}\big(-K_X-vF\big)dv+\int_1^2(2-u)^3\cdot\frac{1}{3}\int_0^\infty\mathrm{vol}\big(-K_X-wF\big)dwdu\right)=\\
&=\frac{3}{4}\left(S_X(F)+\frac{1}{4}\cdot S_X(F)\right)=\frac{15}{16}S_X(F)\le\frac{15}{16}\cdot\frac{A_X(F)}{\delta_Q(X)}.
\end{aligned}$$

We see that $\delta_Q(\mathbf{X})\ge\frac{16}{15}\delta_Q(X)$. Thus, by Main Theorem if every fiber of p_1 has at most $\mathbb{A}_2$ singularities the result follows. □

10.2.3 Family №2.2

Let R be a surface with isolated singularities of degree $(2,4)$ in $\mathbb{P}^1\times\mathbb{P}^2$, let $\pi:\mathbf{X}\to\mathbb{P}^1\times\mathbb{P}^2$ be a double cover ramified over the surface R. Let $pr_1:\mathbb{P}^1\times\mathbb{P}^2\to\mathbb{P}^1$ be the projection on the first factor. Set $p_1=pr_1\circ\pi$. Then p_1 is a fibration into del Pezzo surfaces of degree 2.

Remark 10.2.3*.* Every smooth element in Family №2.2. is known to be K-stable by Cheltsov et al. (2024).

Main Theorem gives the following corollary:

COROLLARY 7. If every fiber X of p_1 has at most $\mathbb{A}_3$ singularities, then $\mathbf{X}$ is K-stable.

Proof. If X is an irreducible fiber of p_1 then arguing as in the proof of (Fujita, 2016, Theorem 10.1) we have $S_{\mathbf{X}}(X)<1$. The divisor $-K_{\mathbf{X}}-uX$ is nef if and only if $u\le 1$ and the Zariski Decomposition is given by $P(u)=-K_{\mathbf{X}}-uX$ and $N(u)=0$ for $u\in[0,1]$. By (Araujo et al., 2023, Corollary 1.110) for any divisor F such that $Q\in C_X(F)$ over X we get:

$$S\big(W^X_{\bullet,\bullet};F\big)=$$

$$= \frac{3}{(-K_{\mathbf{X}})^3}\left(\int_0^{\tau}\big(P(u)^2\cdot X\big)\cdot \mathrm{ord}_Q\Big(N(u)\big|_X\Big)du+\int_0^{\tau}\int_0^{\infty}\mathrm{vol}\big(P(u)\big|_X-vF\big)dvdu\right)=$$
$$=\frac{3}{6}\int_0^{\tau}\int_0^{\infty}\mathrm{vol}\big(P(u)\big|_X-vF\big)dvdu=\frac{3}{6}\int_0^{1}\int_0^{\infty}\mathrm{vol}\big(-K_X-vF\big)dvdu=$$
$$=\frac{3}{6}\int_0^{\infty}\mathrm{vol}\big(-K_X-vF\big)dv=\frac{1}{2}\int_0^{\infty}\mathrm{vol}\big(-K_X-vF\big)dv=$$
$$=S_X(F)\le\frac{A_X(F)}{\delta_Q(X)}.$$

We see that $\delta_Q(\mathbf{X})\ge\delta_Q(X)$. Thus, by Main Theorem if every fiber of p_1 has at most $\mathbb{A}_3$ singularities, i.e. all the fibers are K-semi-stable, then $\delta(\mathbf{X})>1$ since if Q is a singular point and X is a fiber containing Q, such that $\delta_Q(X)=1$ then $\frac{A_{\mathbf{X}}(\mathbf{E})}{S_{\mathbf{X}}(\mathbf{E})}>\min\left\{\frac{1}{S_{\mathbf{X}}(X)},\delta_Q\big(X,W^X_{\bullet,\bullet}\big)\right\}$ from (Araujo et al., 2023, Corollary 1.108.) and otherwise $\delta_Q(X)>1$ and the result follows. □

10.2.4 Family №1.12 (Del Pezzo Threefolds of degree 2)

Let $\psi:\mathbf{V}\to\mathbb{P}^3$ be the double cover branched along the reduced possibly reducible quartic surface R. Set $H=\psi^*(\mathscr{O}_{\mathbb{P}^3}(1))$. Then $\mathbf{V}$ is a del Pezzo threefold of degree 2. One can show that a general surface in $|H|$ is a smooth del Pezzo surface of degree 2.

Remark 10.2.4. Every smooth element in Family №1.12. is known to be K-stable by Araujo et al. (2023) and Dervan (2016).

Singular Del Pezzo Threefolds of degree 2 were studied in Ascher et al. (2023). It follows from Ascher et al. (2023); Shah (1981) that $\mathbf{V}$ is K-polystable if and only if the quartic surface R is GIT-polystable with respect to natural action $PGL(4)$ except for those of the form $(x_0x_2+x_1^2+x_3^2)^2+a\cdot x_3^4$ for $a\in\mathbb{C}$. Main Theorem gives the following (slightly weaker) corollary:

COROLLARY 3. If R has $\mathbb{A}_n$-singulalrities, then $\mathbf{V}$ is K-stable.

Proof. Suppose X is an irreducible element of $|H|$ then $S_{\mathbf{V}}(X)<1$. As for Family №2.2 we fix a prime divisor $\mathbf{E}$ over $\mathbf{V}$. Then we set $Z=C_{\mathbf{V}}(\mathbf{E})$ and if $\beta(\mathbf{E})\leqslant 0$, then $\delta_Q(X,W^X_{\bullet,\bullet})\leqslant 1$. Let Q be a general point in Z, Let X be the general element of $|H|$ that contains Q. The divisor $-K_{\mathbf{V}}-uX$ is nef if and only if $u\le 2$ and the Zariski Decomposition is given by by $P(u)=-K_{\mathbf{V}}-uX\sim(2-u)X$ and $N(u)=0$ for $u\in[0,2]$. By (Araujo et al., 2023, Corollary 1.110) for any divisor F such that $Q\in C_X(F)$ over X we get:

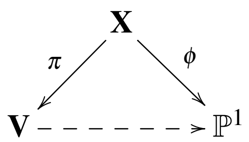
$$S\big(W^X_{\bullet,\bullet};F\big)=$$
$$\frac{3}{(-K_{\mathbf{V}})^3}\left(\int_0^{\tau}\big(P(u)^2\cdot X\big)\cdot \mathrm{ord}_Q\Big(N(u)\big|_X\Big)du+\int_0^{\tau}\int_0^{\infty}\mathrm{vol}\big(P(u)\big|_X-vF\big)dvdu\right)=$$
$$=\frac{3}{16}\int_0^{\tau}\int_0^{\infty}\mathrm{vol}\big(P(u)\big|_X-vF\big)dvdu=\frac{3}{16}\int_0^{2}(2-u)^3\int_0^{\infty}\mathrm{vol}\big(-K_X-wF\big)dwdu=$$

$$= \frac{3}{8}\int_0^2 (2-u)^3 \Big(\frac{1}{2}\int_0^\infty \mathrm{vol}\big(-K_X - wF\big)dw\Big)du = \frac{3}{8}\int_0^2 (2-u)^3 S_X(F)du =$$
$$= \frac{3}{2}S_X(F) \leq \frac{3}{2}\frac{A_X(F)}{\delta_Q(X)}.$$

We get that $\delta_Q(\mathbf{V}) \geq \frac{2}{3}\delta_Q(X)$. For X with at most $\mathbb{A}_1$-singularities we have $\delta_Q(X) \geq \frac{3}{2}$. If Q is a singular point and there exists an element X of $|H|$ with $\delta_Q(X) = \frac{3}{2}$ then $\frac{A_{\mathbf{X}}(\mathbf{E})}{S_{\mathbf{X}}(\mathbf{E})} > \min\left\{\frac{1}{S_{\mathbf{X}}(X)}, \delta_Q\big(X, W^X_{\bullet,\bullet}\big)\right\}$ from (Araujo et al., 2023, Corollary 1.108.) and otherwise we choose X with $\delta_Q(X) > \frac{3}{2}$ so $\delta_Q(\mathbf{V}) > 1$ if X has at most $\mathbb{A}_1$-singularities. This is the case when R has $\mathbb{A}_n$-singularities and the result follows. □

10.2.5 Family №2.3

Let $\psi : \mathbf{V} \to \mathbb{P}^3$ be the double cover branched along the reduced possibly reducible quartic surface R. Set $H = \psi^*(\mathscr{O}_{\mathbb{P}^3}(1))$. Let S_1 and S_2 be two distinct surfaces in the linear system $|H|$, and let $\mathscr{C} = S_1 \cap S_2$. Suppose that the curve $\mathscr{C}$ is smooth. Then $\mathscr{C}$ is an elliptic curve by the adjunction formula. Let $\pi : \mathbf{X} \to \mathbf{V}$ be the blow up of the curve $\mathscr{C}$, and let E be the π-exceptional surface. We have the following commutative diagram:



Where $\mathbf{V} \dashrightarrow \mathbb{P}^1$ is the rational map given by the pencil that is generated by S_1 and S_2, and ϕ is a fibration into del Pezzo surfaces of degree 2.

Remark 10.2.5. Every smooth threefold in Family №2.3. is known to be K-stable by Cheltsov et al. (2024).

Main Theorem gives the following corollary:

COROLLARY 4. If every fiber X of ϕ at most $\mathbb{A}_3$ singularities, then $\mathbf{X}$ is K-stable.

Proof. If X is an irreducible fiber of p_1 then arguing as in the proof of (Fujita, 2016, Theorem 10.1) we have $S_{\mathbf{X}}(X) < 1$. As for Family №2.2 we fix a prime divisor $\mathbf{E}$ over $\mathbf{X}$. Then we set $Z = C_{\mathbf{X}}(\mathbf{E})$. Let Q be the point on Z. let X be the fiber of ϕ that passes through Q. Then $-K_{\mathbf{X}} - uX$ is nef if and only if $u \leq 2$ and the Zariski Decomposition is given by

$$P(u) = \begin{cases} -K_{\mathbf{X}} - uX \sim (2-u)X + E \text{ if } u \in [0,1], \\ -K_{\mathbf{X}} - uX - (u-1)E \sim (2-u)\pi^*(H) \text{ if } u \in [1,2]. \end{cases}$$

$$N(u) = \begin{cases} 0 \text{ if } u \in [0,1], \\ (u-1)E \text{ if } u \in [1,2]. \end{cases}$$

We apply Abban-Zhuang method to prove that $Q \notin E \cong \mathscr{C} \times \mathbb{P}^1$. By (Araujo et al., 2023, Corollary 1.110) for any divisor F such that $Q \in C_X(F)$ over X we get:

$$
\begin{aligned}
&S(W^X_{\bullet,\bullet};F) = \\
&= \frac{3}{(-K_{\mathbf{X}})^3}\Bigg(\int_0^\tau \big(P(u)^2\cdot X\big)\cdot \mathrm{ord}_Q\Big(N(u)\big|_X\Big)du + \int_0^\tau\int_0^\infty \mathrm{vol}\big(P(u)\big|_X - vF\big)dvdu\Bigg) = \\
&= \frac{3}{8}\int_0^\tau\int_0^\infty \mathrm{vol}\big(P(u)\big|_X - vF\big)dvdu = \\
&= \frac{3}{8}\Bigg(\int_0^1\int_0^\infty \mathrm{vol}\big(-K_X - vF\big)dvdu + \int_1^2\int_0^\infty \mathrm{vol}\big(-K_X-(u-1)E|_X - vF\big)dvdu\Bigg) = \\
&= \frac{3}{8}\Bigg(\int_0^\infty \mathrm{vol}\big(-K_X - vF\big)dv + \int_1^2 (2-u)^3\int_0^\infty \mathrm{vol}\big(-K_X - wF\big)dwdu\Bigg) = \\
&= \frac{3}{4}\Bigg(\frac{1}{2}\int_0^\infty \mathrm{vol}\big(-K_X - vF\big)dv + \int_1^2 (2-u)^3\cdot\frac{1}{2}\int_0^\infty \mathrm{vol}\big(-K_X - wF\big)dwdu\Bigg) = \\
&= \frac{3}{4}\Bigg(S_X(F) + \frac{1}{4}\cdot S_X(F)\Bigg) = \frac{15}{16}S_X(F) \leq \frac{15}{16}\cdot\frac{A_X(F)}{\delta_Q(X)}.
\end{aligned}
$$

We see that $\delta_Q(\mathbf{X}) \geq \frac{16}{15}\delta_Q(X)$. Thus, by Main Theorem if every fiber of p_1 has at most $\mathbb{A}_3$ singularities the result follows. □

10.2.6 Family №1.11 (Del Pezzo Threefolds of degree 1)

Let $\mathbf{V}$ be a Fano threefold with canonical Gorenstein singularities such that $-K_{\mathbf{V}} \sim 2H$ for some $H \in \mathrm{Pic}(\mathbf{V})$ with $H^3 = 1$. Then $\mathbf{V}$ is a sextic hypersurface in $\mathbb{P}(1,1,1,2,3)$ and a del Pezzo threefold of degree 1. A general element in $|H|$ is a Du Val del Pezzo surface of degree 1 and if $\mathbf{V}$ has isolated singularities then a general surface in $|H|$ is a smooth.

Remark 10.2.6. Every smooth element in Family №1.11. is K-stable by Araujo et al. (2023).

Main Theorem gives the following corollary:

COROLLARY 5. Suppose that for any point Q on $\mathbf{V}$ there exists an element $X \in |H|$ such that $Q \in X$ and X has at most $\mathbb{A}_2$ singularities then $\mathbf{V}$ is K-stable.

Proof. Suppose X is an irreducible element of $|H|$ then $S_{\mathbf{V}}(X) < 1$. As explained above we fix a prime divisor $\mathbf{E}$ over $\mathbf{V}$. Then we set $Z = C_{\mathbf{V}}(\mathbf{E})$ and if $\beta(\mathbf{E}) \leqslant 0$, then $\delta_Q(X, W^X_{\bullet,\bullet}) \leqslant 1$. Let Q be a general point in Z, Let X be the general element of $|H|$ that contains Q. The divisor $-K_{\mathbf{V}} - uX$ is nef if and only if $u \leq 2$ and the Zariski Decomposition is given by by $P(u) = -K_{\mathbf{V}} - uX \sim (2-u)X$ and $N(u) = 0$ for $u \in [0,2]$. By (Araujo et al., 2023, Corollary 1.110) for any divisor F such that $Q \in C_X(F)$ over X we get:

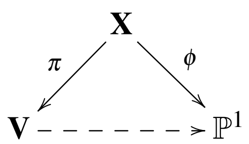
$$S(W^X_{\bullet,\bullet};F) =$$

$$= \frac{3}{(-K_{\mathbf{V}})^3}\left(\int_0^\tau \big(P(u)^2\cdot X\big)\cdot \mathrm{ord}_Q\Big(N(u)\big|_X\Big)du + \int_0^\tau\int_0^\infty \mathrm{vol}\big(P(u)\big|_X - vF\big)dvdu\right) =$$
$$= \frac{3}{8}\int_0^\tau\int_0^\infty \mathrm{vol}\big(P(u)\big|_X - vF\big)dvdu = \frac{3}{8}\int_0^2 (2-u)^3\int_0^\infty \mathrm{vol}\big(-K_X - wF\big)dwdu =$$
$$= \frac{3}{8}\int_0^2 (2-u)^3\Big(\int_0^\infty \mathrm{vol}\big(-K_X - wF\big)dw\Big)du = \frac{3}{8}\int_0^2 (2-u)^3 S_X(F)du =$$
$$= \frac{3}{2}S_X(F) \leq \frac{3}{2}\frac{A_X(F)}{\delta_Q(X)}.$$

We get that $\delta_Q(\mathbf{V}) \geq \frac{2}{3}\delta_Q(X)$. For X with at most $\mathbb{A}_2$-singularities we have $\delta_Q(X) \geq \frac{3}{2}$. If Q is a singular point and there exists an element X of $|H|$ with $\delta_Q(X) = \frac{3}{2}$ then $\frac{A_{\mathbf{X}}(\mathbf{E})}{S_{\mathbf{X}}(\mathbf{E})} > \min\left\{\frac{1}{S_{\mathbf{X}}(X)}, \delta_Q\big(X, W^X_{\bullet,\bullet}\big)\right\}$ from (Araujo et al., 2023, Corollary 1.108.) and otherwise we choose X with $\delta_Q(X) > \frac{3}{2}$ so $\delta_Q(\mathbf{V}) > 1$ if X has at most $\mathbb{A}_2$-singularities and the result follows. □

10.2.7 Family №2.1

Let $\mathbf{V}$ be a Fano threefold with canonical Gorenstein singularities such that $-K_{\mathbf{V}} \sim 2H$ for some $H \in \mathrm{Pic}(\mathbf{V})$ with $H^3 = 1$. Then $\mathbf{V}$ is a sextic hypersurface in $\mathbb{P}(1,1,1,2,3)$ and a del Pezzo threefold of degree 1. Let S_1 and S_2 be two distinct surfaces in the linear system $|H|$, and let $\mathscr{C} = S_1 \cap S_2$. Suppose that the curve $\mathscr{C}$ is smooth. Then $\mathscr{C}$ is an elliptic curve by the adjunction formula. Let $\pi : \mathbf{X} \to \mathbf{V}$ be the blow up of the curve $\mathscr{C}$, and let E be the π-exceptional surface. We have the following commutative diagram:



Where $\mathbf{V} \dashrightarrow \mathbb{P}^1$ is the rational map given by the pencil that is generated by S_1 and S_2, and ϕ is a fibration into del Pezzo surfaces of degree 1.

Remark 10.2.7. Every smooth Fano threefold in Family №2.1. is K-stable by Cheltsov et al. (2024).

Main Theorem gives the following corollary:

COROLLARY 6. If every fiber X of ϕ at most $\mathbb{D}_4$ singularities, then $\mathbf{X}$ is K-stable.

Proof. If X is an irreducible fiber of p_1 then we have $S_{\mathbf{X}}(X) < 1$. We now fix a prime divisor $\mathbf{E}$ over $\mathbf{X}$. Then we set $Z = C_{\mathbf{X}}(\mathbf{E})$. Let Q be the point on Z. let X be the fiber of ϕ that passes through Q. Then $-K_{\mathbf{X}} - uX$ is nef if and only if $u \leq 2$ and the Zariski Decomposition is given by

$$P(u) = \begin{cases} -K_{\mathbf{X}} - uX \sim (2-u)X + E \text{ if } u \in [0,1], \\ -K_{\mathbf{X}} - uX - (u-1)E \sim (2-u)\pi^*(H) \text{ if } u \in [1,2]. \end{cases}$$

$$N(u) = \begin{cases} 0 \text{ if } u \in [0,1], \\ (u-1)E \text{ if } u \in [1,2]. \end{cases}$$

We apply Abban-Zhuang method to prove that $Q \notin E \cong \mathscr{C} \times \mathbb{P}^1$. By (Araujo et al., 2023, Corollary 1.110) for any divisor F such that $Q \in C_X(F)$ over X we get:

$$\begin{aligned} & S\big(W^X_{\bullet,\bullet};F\big) = \\ & = \frac{3}{(-K_{\mathbf{X}})^3}\left(\int_0^\tau \big(P(u)^2 \cdot X\big) \cdot \mathrm{ord}_Q\Big(N(u)\big|_X\Big) du + \int_0^\tau \int_0^\infty \mathrm{vol}\big(P(u)\big|_X - vF\big) dv du\right) = \\ & = \frac{3}{4}\int_0^\tau \int_0^\infty \mathrm{vol}\big(P(u)\big|_X - vF\big) dv du = \\ & = \frac{3}{4}\left(\int_0^1 \int_0^\infty \mathrm{vol}\big(-K_X - vF\big) dv du + \int_1^2 \int_0^\infty \mathrm{vol}\big(-K_X - (u-1)E|_X - vF\big) dv du\right) = \\ & = \frac{3}{4}\left(\int_0^\infty \mathrm{vol}\big(-K_X - vF\big) dv + \int_1^2 (2-u)^3 \int_0^\infty \mathrm{vol}\big(-K_X - wF\big) dw du\right) = \\ & = \frac{3}{4}\left(\int_0^\infty \mathrm{vol}\big(-K_X - vF\big) dv + \int_1^2 (2-u)^3 \int_0^\infty \mathrm{vol}\big(-K_X - wF\big) dw du\right) = \\ & = \frac{3}{4}\left(S_X(F) + \frac{1}{4} \cdot S_X(F)\right) = \frac{15}{16} S_X(F) \leq \frac{15}{16} \cdot \frac{A_X(F)}{\delta_Q(X)}. \end{aligned}$$

We see that $\delta_Q(\mathbf{X}) \geq \frac{16}{15}\delta_Q(X)$. Thus, by Main Theorem if every fiber of p_1 has at most $\mathbb{D}_4$ singularities the result follows. □

Chapter 11

K-stability classification for Family №3.12

In this chapter we find all K-polystable smooth Fano threefolds that can be obtained as blowup of $\mathbb{P}^3$ along the disjoint union of a twisted cubic curve and a line. The results presented in this chapter were published in *Journal of London Mathematical Society* (see Denisova (2024b)).

11.1 Calabi Problem for Family №3.12

By the works of Chen-Donaldson-Sun and Tian (see Chen et al. (2015) and Tian (2015)), a Fano manifold admits a Kähler–Einstein metric if and only if it is K-polystable. For two-dimensional Fano varieties (del Pezzo surfaces) Tian and Yau proved that a smooth del Pezzo surface is K-polystable if and only if it is not a blow up of $\mathbb{P}^2$ in one or two points (see Tian (1990); Tian and Yau (1987)). For three-dimensional Fano varieties the situation is more challenging. Smooth Fano threefolds over the field $\mathbb{C}$ have been classified in Iskovskikh (1997, 1998); Mori and Mukai (1981, 2003) into 105 families. The detailed description of these families can be found in Araujo et al. (2023) where the following problem was posed:

Calabi Problem. *Find all K-polystable smooth Fano threefolds in each family.*

In this chapter, we will solve the Calabi problem completely for Family №3.12, by using Abban–Zhuang theory (see Abban and Zhuang (2022)), combined with previous partial results on this family presented in (Araujo et al., 2023, §5.18)

Suppose X is a member of Family №3.12. Then we describe X as the blowup $\pi : X \to \mathbb{P}^3$ of $\mathbb{P}^3$ at a twisted cubic C and line L that is disjoint from C (see Section 11.3 for an explicit description of all members of this family). The deformation Family №3.12 contains a unique smooth Fano threefold with an infinite automorphism group. It was proven in Araujo et al. (2023) that this smooth Fano threefold is K-polystable (the proof heavily relies on the automorphism group of this threefold). Using this, the authors of Araujo et al. (2023) described a strictly K-semistable smooth Fano threefold in this family. Moreover, it has been conjectured in Araujo et al. (2023) that all other smooth Fano threefolds in the deformation Family №3.12 are K-stable. In this article we proved this conjecture. This completely solves the K-stability problem for smooth Fano threefolds in the deformation Family №3.12.

MAIN THEOREM 3. All the smooth threefolds except one in Family №3.12 are K-polystable.

Hence, all smooth Fano threefolds in Family №3.12 except one described in (Araujo et al., 2023, §7.7) admit a Kähler–Einstein metric.

11.2 Preliminary results

Let X be a Fano variety with Kawamata log terminal singularities, let G be a reductive subgroup in $\mathrm{Aut}(X)$, let $f\colon \widetilde{X} \to X$ be a G-equivariant birational morphism, let F be a G-invariant prime divisor in $\widetilde{X}$, and let $n = \dim(X)$.

Definition 11.2.1. We say that F is a G-invariant prime divisor *over* the Fano variety X. If F is f-exceptional, we say that F is an exceptional G-invariant prime divisor *over* X. We will denotes the subvariety $f(F)$ by $C_X(F)$.

Let

$$S_X(F) = \frac{1}{(-K_X)^n} \int_0^{\tau} \mathrm{vol}(f^*(-K_X) - uF)du,$$

where $\tau = \tau(F)$ is the pseudo-effective threshold of F with respect to $-K_X$, i.e. we have

$$\tau(F) = \sup\Big\{u \in \mathbb{Q}_{>0} \ \Big|\ f^*(-K_X) - uF \text{ is big}\Big\}. \tag{11.2.2}$$

Let $\beta(F) = A_X(F) - S_X(F)$, where $A_X(F)$ is the log discrepancy of the divisor F.

Theorem 11.2.3 (Corollary 4.14 in Zhuang (2021))**.** *Suppose that $\beta(F) > 0$ for every G-invariant prime divisor F over X. Then X is K-polystable.*

Theorem 11.2.4 (Theorem 10.1 in Fujita (2016))**.** *Let X be any smooth Fano threefold that is not contained in the following 41 deformation families:*

№1.17, №2.23, №2.26, №2.28, №2.30, №2.31, №2.33, №2.34, №2.35, №2.36, №3.9, №3.14, №3.16, №3.18, №3.19, №3.21, №3.22, №3.23, №3.24, №3.25, №3.26, №3.28, №3.29, №3.30, №3.31, №4.2, №4.4, №4.5, №4.7, №4.8, №4.9, №4.10, №4.11, №4.12, №5.2, №5.3, №6.1, №7.1, №8.1, №9.1, №10.1.

Then $S_X(Y) < 1$ for every irreducible surface $Y \subset X$, i.e. X is divisorially stable.

Theorem 11.2.5 (Collolary 1.110 in Araujo et al. (2023))**.** *Let X be a smooth Fano threefold, let Y be an irreducible normal surface in the threefold X, let Z be an irreducible curve in Y, and let F be a prime divisor over the threefold X such that $C_X(F) = Z$. Then*

$$\frac{A_X(F)}{S_X(F)} \geqslant \min\left\{\frac{1}{S_X(Y)}, \frac{1}{S\big(W^Y_{\bullet,\bullet}; Z\big)}\right\} \tag{11.2.6}$$

and

$$S(W^Y_{\bullet,\bullet};Z) = \frac{3}{(-K_X)^3}\int_0^\tau \big(P(u)^2\cdot Y\big)\cdot \mathrm{ord}_Z\Big(N(u)\big|_Y\Big)du + \\ + \frac{3}{(-K_X)^3}\int_0^\tau\int_0^\infty \mathrm{vol}\big(P(u)\big|_Y - vZ\big)dvdu,$$

where $P(u)$ is the positive part of the Zariski decomposition of the divisor $-K_X - uY$, and $N(u)$ is its negative part.

Lemma 11.2.7 (Lemma 1.44 in Araujo et al. (2023)). *Let X be a smooth Fano variety, Z be a proper irreducible subvariety in X with $\dim(Z) \geqslant 1$. Let $f\colon \widetilde{X} \to X$ be an arbitrary G-equivariant birational morphism, let F be a G-invariant prime divisor in $\widetilde{X}$ such that $Z \subseteq f(F)$, and let $\tau(F)$ satisfy* (11.2.2)*. Then*

$$\frac{A_X(F)}{S_X(F)} > \frac{n+1}{n}\alpha_{G,Z}(X),$$

where

$$\alpha_{G,Z}(X) = \sup\left\{\lambda\in\mathbb{Q}\ \middle|\ \begin{array}{l}\text{the pair } (X,\lambda D) \text{ is log canonical at general point of } Z \text{ for any}\\ \text{effective } G\text{-invariant } \mathbb{Q}\text{-divisor } D \text{ on } X \text{ such that } D\sim_{\mathbb{Q}} -K_X\end{array}\right\}.$$

Lemma 11.2.8 (Corollary A.13 in Araujo et al. (2023)). *Suppose $X = \mathbb{P}^3$ and $B_X \sim_{\mathbb{Q}} -\lambda K_X$ for some rational number $\lambda < \frac{3}{4}$. Let Z be the union of one-dimensional components of $\mathrm{Nklt}(X,B_X)$. Then $\mathscr{O}_{\mathbb{P}^3}(1)\cdot Z \leqslant 1$. In particular, if $Z \neq 0$, then Z is a line.*

Lemma 11.2.9 (Corollary A.15 in Araujo et al. (2023)). *Suppose that X is a smooth Fano threefold, $B_X \sim_{\mathbb{Q}} -\lambda K_X$ for some rational number $\lambda < 1$, and there exists a surjective morphism with connected fibers $\phi\colon X \to \mathbb{P}^1$. Set $H = \phi^*(\mathscr{O}_{\mathbb{P}^1}(1))$. Let Z be the union of one-dimensional components of $\mathrm{Nklt}(X,\lambda B_X)$. Then $H\cdot Z \leqslant 1$.*

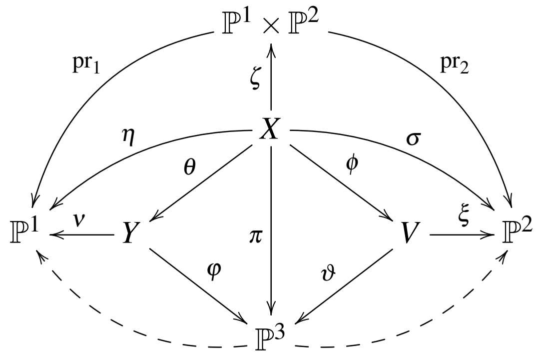
11.3 Geometry of Fano Threefolds in Family №3.12

11.3.1 Basic properties

Let C be the smooth twisted cubic curve in $\mathbb{P}^3$ that is the image of the map $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by

$$[x:y] \to [x^3 : x^2y : xy^2 : y^3]$$

let L be a line in $\mathbb{P}^3$ that is disjoint from C, and let $\pi : X \to \mathbb{P}^3$ be the blow up of $\mathbb{P}^3$ along C and L. Then X is *a Fano threefold in Family №3.12* and all threefolds in this family can be obtained this way. Note that there exists the following commutative diagram:



Where:

- φ is the blowup of a line L,
- ϑ is the blowup of a curve C,
- ϕ is the blowup of a curve ϑ^*L,
- θ is the blowup of a curve φ^*C,
- the left dashed arrow is the linear projection from the line L,
- the right dashed arrow is given by the linear system of quadrics that contain C,
- ξ is a $\mathbb{P}^1$-bundle,
- ν is a $\mathbb{P}^2$-bundle,
- σ is a non-standard conic bundle,
- η is a fibration into the del Pezzo surfaces of degree 6,
- ζ is the contraction of the proper transforms of the quartic surface in $\mathbb{P}^3$ that is spanned by the secants of the curve C that intersect L,
- pr_1 and pr_2 are projections to the first and the second factors, respectively.

Let H be a plane in $\mathbb{P}^3$, E_L be the exceptional surface of π that is mapped to L, E_C be the exceptional surface of π that is mapped to C, R be ζ-exceptional surface. Then

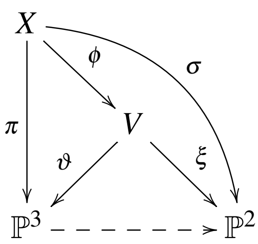
$$R \sim_{\mathbb{Q}} \pi^*(4H) - 2E_C - E_L,$$

and

$$-K_X \sim_{\mathbb{Q}} \pi^*(4H) - E_C - E_L.$$

11.3.2 Construction of R

Consider the commutative diagram:



Where ξ is a $\mathbb{P}^1$-bundle given by the linear system $|\vartheta^*(2H) - E_C|$, ϑ is the blowup of C and the dashed arrow is given by the linear system of quadrics containing C, ϕ is the blowup of $\vartheta^* L$. Denote $\tilde{L} = \vartheta^* L$. What is the image of $\tilde{L}$ in $\mathbb{P}^2$? We have that

$$\tilde{L} \cdot (\vartheta^*(2H) - E_C) = 2$$

which means that $\xi(\tilde{L})$ is a conic. The preimage of this conic on V is spanned by strict transforms of secants of C which intersect $\tilde{L}$. Therefore, $\pi(R)$ is spanned by secants of C that intersect L. Note that the class of the preimage is

$$\xi^*(\mathscr{O}_{\mathbb{P}^2}(2)) = 2(\vartheta^*(2H) - E_C) = \vartheta^*(4H) - 2E_C.$$

Moreover $\xi(\tilde{L})$ is a smooth conic and ξ is a $\mathbb{P}^1$-bundle thus the preimage of $\xi(\tilde{L})$ is a smooth surface so it is smooth along $\tilde{L}$ thus the class of R in $\mathbb{P}^3$ is given by

$$R \sim_{\mathbb{Q}} \pi^*(4H) - 2E_C - E_L.$$

11.3.3 $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ - action on X

Note that $\mathrm{Aut}(X) \cong \mathrm{Aut}(\mathbb{P}^3, C + L)$. On the other hand, we have

$$\mathrm{Aut}(\mathbb{P}^3, C) = \mathrm{PGL}_2(\mathbb{C}),$$

where $\mathrm{Aut}(\mathbb{P}^3, C)$ is the group of automorphisms of $\mathbb{P}^3$ which fix C as a set.

Types of threefolds in Family 3.12

We look at the projection from the line L which is disjoint from C to $\mathbb{P}^1$:

$$\phi_L : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1,$$

which gives a 3-cover of $\mathbb{P}^1$:

$$\phi_L|_C : C \xrightarrow{3:1} \mathbb{P}^1.$$

By Riemann-Hurwitz we have that the degree of the ramification divisor is 4. The multiplicity in each ramification point is either 2 or 3 so we have three options:

- there are two ramification points both of multiplicity 3,
- there is one ramification point of multiplicity 3 and two ramification points of multiplicity 2,
- there are four ramification points of multiplicity 2.

We see that there are at least two ramification points on C. By acting on C by the $\mathrm{PGL}(2,\mathbb{C})$ we can make these points to be $p_1 = [1:0]$, $p_2 = [0:1]$ on C. Now we look at the line L. It is the intersection of 2 planes which are tangent to C at points p_1 and p_2 (note that these planes are different since the plane intersects the cubic C in three points, so the same plane cannot be tangent to C at two points) so it is given by the equations:

$$L : \begin{cases} x_0 = r_1 x_1, \\ x_3 = r_2 x_2. \end{cases}$$

We have 3 cases:
1. $r_1 = r_2 = 0$ so L is given by the equations:

$$L : \begin{cases} x_0 = 0, \\ x_3 = 0. \end{cases}$$

Here we have two ramification points of multiplicity 3. This case was described in Araujo et al. (2023). The corresponding threefold X is K-polystable in this case.
2. $r_1 = 0$, $r_2 \neq 0$ (which is symmetric to the case $r_1 \neq 0$, $r_2 = 0$) so L is given by the equations:

$$L : \begin{cases} x_0 = 0, \\ x_3 = r_2 x_2. \end{cases}$$

Using the action of $\mathbb{C}^*$ by the matrix which fixes C:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_2^2 & 0 \\ 0 & 0 & 0 & r_2^3 \end{pmatrix}$$

We can assume that L is given by

$$L: \begin{cases} x_0 = 0, \\ x_3 = x_2. \end{cases}$$

Here we have one ramification point of multiplicity 3 and two ramification points of multiplicity 2. This case was described in Araujo et al. (2023) where it was proven that X is not K-polystable.

3. $r_1 \neq 0$, $r_2 \neq 0$ so L is given by the equations

$$L: \begin{cases} x_0 = r_1 x_1, \\ x_3 = r_2 x_2. \end{cases}$$

Using the action of $\mathbb{C}^*$ by the matrix which fixes C:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & \lambda^3 \end{pmatrix},$$

where λ satisfies $\lambda^2 = \frac{r_2}{r_1}$. We can assume that L is given by

$$L: \begin{cases} x_0 = r x_1, \\ x_3 = r x_2. \end{cases}$$

Note that:

- $r \neq 0$ since otherwise we are in case **1**.
- $r \neq \pm 1$ since otherwise L intersects C which is prohibited,
- $r \neq \pm 3$ since otherwise there exists a plane containing L which is tangent to C with multiplicity 3 (it is a plane given by $x_3 + 3x_3 + 3x_1 + x_0 = 0$ in case $r = -3$ and a plane $-x_3 + 3x_3 - 3x_1 + x_0 = 0$ in case $r = 3$) so this case is projectively isomorphic to the case **2**.

Now the involution on $\mathbb{P}^3$ given by $[x_0 : x_1 : x_2 : x_3] \to [x_3 : x_2 : x_1 : x_0]$ fixes C an L. We can construct a similar involution for any pair of four ramification points on $C \cong \mathbb{P}^1$. This gives the action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. More precisely this group is generated by the involutions viewed on $\mathbb{P}^1$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & -\frac{r(r^2-5+q)}{2(r^2-3+q)} \\ \frac{r^2+3+q}{4r} & -1 \end{pmatrix}$$

where q is any root of the equation $r^4 - 10r^2 + 9$. The action on $\mathbb{P}^3$ is given by the map induced by $[x : y] \to [x^3 : x^2y : xy^2 : y^2]$.

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ - fixed points on X

From now on we assume until the end of this section that we are in case **3** of the previous part and $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In particular, $\mathrm{Aut}(X)$ is finite (see Cheltsov, Przyjalkowski, and Shramov (2019)).

Lemma 11.3.1. *There are no G-invariant planes on $\mathbb{P}^3$*

Proof. Note that $G \hookrightarrow \mathrm{Aut}(C)$ since C is a spatial curve. If there exists a G-invariant plane Π consider the intersection of Π with C. There are three points in $\Pi \cap C$ counted with multiplicities. Thus, since the order of G is 4 then there is a G-fixed point on $C \cong \mathbb{P}^1$, which is a contradiction. □

Corollary 11.3.2. *There are no G-fixed points on $\mathbb{P}^3$.*

Corollary 11.3.3. *The threefold X does not contain G-invariant points.*

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ - invariant Quadrics Containing C

Let $\mathscr{M}$ be the linear system of quadrics on $\mathbb{P}^3$ that contain the curve C.

Lemma 11.3.4. *The linear system $\mathscr{M}$ is 3-dimensional, it contains exactly 3 G-invariant surfaces, and these surfaces are smooth.*

Proof. Note that this statement does not depend on the equivariant choice of coordinates because $\mathrm{PGL}_2(\mathbb{C})$ contains a unique subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ up to conjugation. So we can choose coordinates such that the generators of our group will look like:

$$\tau_1 : [x : y] \to [y : x],$$

$$\tau_2 : [x : y] \to [x : -y].$$

This gives us the action on $\mathbb{P}^3$ by:

$$\tau_1 : [x_0 : x_1 : x_2 : x_3] \to [x_3 : x_2 : x_1 : x_0],$$

$$\tau_2 : [x_0 : x_1 : x_2 : x_3] \to [x_0 : -x_1 : x_2 : -x_3].$$

The linear system $\mathscr{M}$ is clearly 3-dimensional. We can provide the equations for 3 G-invariant quadrics containing C:

$$Q_1 : x_0x_3 = x_1x_2, \quad Q_2 : x_1^2 + x_2^2 = x_0x_2 + x_1x_3, \quad Q_3 : x_1^2 - x_2^2 = x_0x_2 - x_1x_3.$$

Note that (τ_1, τ_2) acts on the equation of:

- Q_1 by multiplying it by $(1, -1)$,
- Q_2 by multiplying it by $(1, 1)$,
- Q_3 by multiplying it by $(-1, 1)$.

Thus, since the action is pairwise distinct and $\mathscr{M}$ is 3-dimensional there are exactly 3 G-invariant quadrics which we listed above. Note that these quadrics are smooth. □

Now take a G-invariant quadric $Q \in \mathscr{M}$ and look at the intersection of it with L. Note that $L \not\subset Q$ since L does not intersect C. The intersection $Q \cap L$ cannot cannot consist of only one point since we do not have G-fixed points thus $Q \cap L$ consists of two distinct points. These two points do not belong to the same curve of bidegree $(1,0)$ or $(0,1)$ (since these curves are the lines on Q and we know that $L \not\subset Q$). Now we see that the blowup $\tilde{Q} \to Q$ at these points is a del Pezzo surface of degree 6.

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ - invariant lines

Let us describe G-invariant lines in $\mathbb{P}^3$. Assume G is generated by τ_1 and τ_2, as in the proof of Lemma 11.3.4. In this case all G-invariant lines are of the form:

$$\begin{cases} \lambda x_0 + \mu x_2 = 0, \\ \lambda x_3 + \mu x_1 = 0. \end{cases}$$

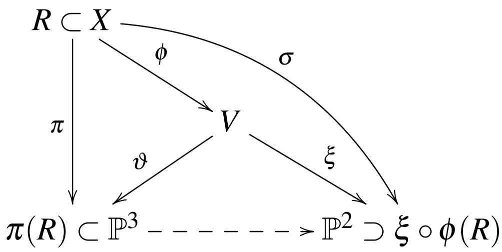
where $[\lambda : \mu] \in \mathbb{P}^1$. All such lines do not intersect each other and lie on the quadric Q_4 given by $x_1x_0 = x_2x_3$. We see that $\mathbb{P}^3$ contains infinitely many G-invariant lines and all of them are contained in Q_4. Among them there are 3 lines that intersect C. We can describe them explicitly. The intersection of this quadric with C consists exactly of 6 points which are:

$$\begin{aligned}
P_1 &= [0:1] = [0:0:0:1], \\
P_2 &= [1:0] = [1:0:0:0], \\
P_3 &= [1:1] = [1:1:1:1], \\
P_4 &= [1:-1] = [1:-1:1:-1], \\
P_5 &= [1:i] = [1:i:-1:-i], \\
P_6 &= [1:-i] = [1:-i:-1:i],
\end{aligned}$$

here in the third column are given the corresponding coordinates on $C \subset \mathbb{P}^3$. Note that τ_1 exchanges P_1 and P_2, τ_2 exchanges P_3 and P_4, as well as τ_2 exchanges P_5 and P_6. Each pair of points belongs to the same line which is different for each point. We denote these lines L_{12}, L_{34}, L_{56}, where L_{ij} is the line connecting points P_i and P_j.

Lemma 11.3.5. *Suppose that Z is a G-invariant irreducible curve on X, $\pi(Z)$ is its image on $\mathbb{P}^3$ and $\pi(Z)$ is a line different from L, L_{12}, L_{34}, L_{56} then $Z \not\subset R$.*

Proof. Suppose Z is contained in R. Consider the following commutative diagram from section 11.3.2:



Where the bottom dashed arrow is given by the linear system of quadrics containing C. Using the equations of quadrics which form the basis of the linear system $\mathscr{M}$ defined in Section 11.3.3 we get the explicit map:

$$\mathbb{P}^3 \dashrightarrow \mathbb{P}^2 \text{ where } [x_0 : x_1 : x_2 : x_3] \dashrightarrow [x_0x_3 - x_1x_2 : x_1^2 - x_0x_2 : x_2^2 - x_1x_3].$$

We know that $\xi \circ \phi(R)$ is a conic. Let's write its equation in $\mathbb{P}^2$ with coordinates $[x : y : z]$:

$$a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2 = 0.$$

We want to look at the preimage of this equation in $\mathbb{P}^3$ which will give the equation for $\pi(R)$. Substituting $[x_0x_3 - x_1x_2 : x_1^2 - x_0x_2 : x_2^2 - x_1x_3]$ into the defining equation of $\xi \circ \phi(R)$ we get:

$$\begin{aligned}\pi(R) : a_1x_3^2x_0^2 - a_2x_2x_3x_0^2 + a_4x_2^2x_0^2 + a_2x_3x_0x_1^2 - 2a_4x_2x_0x_1^2 + a_2x_2^2x_0x_1 + \\ + (-2a_1 + a_5)x_2x_3x_0x_1 - a_3x_3^2x_0x_1 + a_3x_3x_2^2x_0 - a_5x_2^3x_0 + a_4x_1^4 - a_2x_1^3x_2 - a_5x_3x_1^3 + \\ + (a_1 + a_5)x_2^2x_1^2 + a_3x_2x_3x_1^2 + a_6x_3^2x_1^2 - a_3x_2^3x_1 - 2a_6x_2^2x_3x_1 + a_6x_2^4 = 0.\end{aligned}$$

Recall from section 11.3.3 that all G-invariant lines are of the form

$$\begin{cases} \lambda x_0 + \mu x_2 = 0, \\ \lambda x_3 + \mu x_1 = 0. \end{cases}$$

where $[\lambda : \mu] \in \mathbb{P}^1$. Now L is given by

$$L = L_s : \begin{cases} x_0 + s x_2 = 0, \\ x_3 + s x_1 = 0. \end{cases} \quad \text{for some } s \in \mathbb{C}.$$

Note that $s \neq 0$ since otherwise X would have an infinite group of automorphisms. Similarly $\pi(Z)$ is given by

$$\pi(Z) = L_t : \begin{cases} x_0 + t x_2 = 0, \\ x_3 + t x_1 = 0. \end{cases} \quad \text{for } t \in \mathbb{C}.$$

By our assumption L_s is contained in $\pi(R)$. This gives

$$\{a_1 = -1/s,\ a_2 = 0,\ a_3 = 0,\ a_4 = 1,\ a_5 = (s^2+1)/s,\ a_6 = 1\}.$$

So that $\pi(R)$ is given by

$$\pi(R) : x_2^2 x_0^2 s - x_3^2 x_0^2 - 2x_2 x_0 x_1^2 s + (s^2+3) x_2 x_3 x_0 x_1 + (-s^2-1) x_2^3 x_0 + s x_1^4 + \\ + (-s^2-1) x_3 x_1^3 + s^2 x_1^2 x_2^2 + x_3^2 x_1^2 s - 2x_2^2 x_3 x_1 s + s x_2^4 = 0.$$

Similarly since L_t is contained in $\pi(R)$ we get

$$\begin{cases} -t^4 - 4ts + (s^2+3)t^2 + s^2 = 0, \\ s + (-s^2-1)t + t^2 s = 0. \end{cases}$$

and the solution to this system is $s = t$ which means that $L_s = L$ and $L_t = \pi(Z)$ coincide contradicting the assumption on Z. □

11.3.4 Mori Cone $\overline{\mathrm{NE}(X)}$

Let l_L, l_C, l_R be the general fibers of the natural projections $E_L \to L$, $E_C \to C$, $R \to \sigma(R)$. Observe that we can contract any of two rays $\mathbb{R}_{\geqslant 0}[l_L]$, $\mathbb{R}_{\geqslant 0}[l_C]$, $\mathbb{R}_{\geqslant 0}[l_R]$. Indeed l_C and l_L are contracted by $\pi : X \to \mathbb{P}^3$, l_R and l_L are contracted by $\sigma : X \to \mathbb{P}^2$, l_R and l_C are contracted by $\eta : X \to \mathbb{P}^1$. Thus, these curves generate 3 extreme rays $\mathbb{R}_{\geqslant 0}[l_L]$, $\mathbb{R}_{\geqslant 0}[l_C]$, $\mathbb{R}_{\geqslant 0}[l_R]$ of the Mori cone $\overline{\mathrm{NE}(X)}$.

11.3.5 Cone of Effective Divisors $\mathrm{Eff}(X)$

Lemma 11.3.6. *Suppose S is a surface in X then*

$$S \sim a(\pi^*(H) - E_L) + b(2\pi^*(H) - E_C) + cR + eE_L + fE_C,$$

for $a, b, c, e, f \in \mathbb{Z}_{\geqslant 0}$.

Proof. Suppose $\pi(S) \subset \mathbb{P}^3$ is the surface of degree d in $\mathbb{P}^3$. Then we have

$$S \sim d\pi^*(H) - m_L E_L - m_C E_C,$$

where m_L is the multiplicity of $\pi(S)$ in L, m_C is the multiplicity of $\pi(S)$ in C. Suppose that $S \neq E_C$, $S \neq E_L$ and $S \neq R$ for all n. Now let's intersect S with three extreme rays l_L, l_C, l_R corresponding to L, C, R:

- $\pi^*(H) \cdot l_C = 0$,
- $\pi^*(H) \cdot l_L = 0$,
- $\pi^*(H) \cdot l_R = 1$,
- $E_L \cdot l_C = 0$,
- $E_L \cdot l_L = -1$,
- $E_L \cdot l_R = 1$,
- $E_C \cdot l_C = -1$,
- $E_C \cdot l_L = 0$,
- $E_C \cdot l_R = 2$.

So we have that:

$$S \cdot l_C = m_C \geqslant 0, \quad S \cdot l_L = m_L \geqslant 0, \quad S \cdot l_R = d - m_L - 2m_C \geqslant 0.$$

Moreover if l_1 is the general line intersecting L, l_2 is the general secant of C then we get strict inequalities:

$$S \cdot l_1 = d - m_L > 0, \quad S \cdot l_2 = d - 2m_C > 0.$$

Now we want to find the integer positive solutions for:

$$d\pi^*(H) - m_L E_L - m_C E_C = a(\pi^*(H) - E_L) + b(2\pi^*(H) - E_C) + cR + eE_L + fE_C.$$

Comparing the coefficients we get:

$$d = a + 2b + 4c, \quad m_C = b + 2c - f, \quad m_L = a + c - e.$$

The non-negative solution to this system can be given by

$$\begin{cases} a = d - 2m_C, \\ b = m_C, \\ c = 0, \\ e = d - m_L - 2m_C, \\ f = 0. \end{cases}$$

Thus, the cone of effective divisors over $\mathbb{Z}$ is generated by $\pi^*(H)-E_L$, $2\pi^*(H)-E_C$, R, E_L, E_C. □

Corollary 11.3.7. *The cone of effective divisors of X is generated over $\mathbb{Q}$ by $\pi^*(H)-E_L$, R, E_L, E_C. More precisely, suppose S is a surface in X then*

$$S \sim_{\mathbb{Q}} a(\pi^*(H)-E_L)+cR+eE_L+fE_C,$$

for unique $a,c,e,f \in \mathbb{Q}_{\geqslant 0}$.

11.4 Proof of the Main Theorem (Family №3.12)

Suppose X is a member of Family №3.12 such that $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\subset \mathrm{Aut}(X)$. In this section we will prove that X is K-polystable. Then we describe X as the blowup $\pi: X\to\mathbb{P}^3$ of $\mathbb{P}^3$ at a twisted cubic C and line L that is disjoint from C. If X is not K-polystable then it follows from (Zhuang, 2021, Corollary 4.14) that there exists a G-invariant prime divisor F over X such that $\beta(F)=A_X(F)-S_X(F)\leqslant 0$. Let Z be the center of F on X. Then Z is not a point since X has no G-fixed points, and Z is not a surface by (Fujita, 2016, Theorem 10.1), so that Z is a G-invariant irreducible curve.

Lemma 11.4.1. *Suppose that $\pi(Z)\neq L$ then $\pi(Z)$ is not one of the G-invariant lines which does not intersect C.*

Proof. Let's take a G-invariant line $\pi(Z)$ that does not intersect C and consider a general plane H which contains this line. It intersects a line L in one point and a twisted cubic C at three points. Let S be the proper transform of H on X. In this case we have that the induced map $\pi|_S: S\to H$ is the blowup of a plane H in 4 points $b_1=H\cap L$, $b_2,b_3,b_4=H\cap C$. We now need to check that these points are in general position to conclude that S is a del Pezzo surface of degree 5.

To prove that we need to show that the points in $\{b_1,b_2,b_3,b_4\}$ are in general position which means that no three of them belong to the same line. Note that $b_2,b_3,b_4=H\cap C$ do not belong to the same line, because C is an intersection of quadrics. So the only option is that b_1 and two points from the set $\{b_2,b_3,b_4\}$ belong to the same line. Suppose H is a general plane and b_1 and 2 points among $\{b_2,b_3,b_4\}$ are contained in one line ℓ. From Section 11.3.2 we know that $\pi(R)$ is spanned by secants of C that intersect L, so H contains such secant ℓ. Moreover, $\pi(Z)$ intersects ℓ, so we see that $\pi(Z)$ intersects a general secant of C that is contained in $\pi(R)$. Then $Z\subset R$ which contradicts Lemma 11.3.5. So we can choose the hyperplane H in such a way that the points in $\{b_1,b_2,b_3,b_4\}$ are in general position. Thus, S is a del Pezzo surface of degree 5. Let E_1, E_2, E_3, E_4 be the exceptional divisors corresponding to points b_1,b_2,b_3,b_4

	Z	E_1	E_2	E_3	E_4	L_{12}	L_{13}	L_{14}	L_{23}	L_{24}	L_{34}
Z	1	0	0	0	0	1	1	1	1	1	1
E_1	0	−1	0	0	0	1	1	1	0	0	0
E_2	0	0	−1	0	0	1	0	0	1	1	0
E_3	0	0	0	−1	0	0	1	0	1	0	1
E_4	0	0	0	0	−1	0	0	1	0	1	1
L_{12}	1	1	1	0	0	−1	0	0	0	0	1
L_{13}	1	1	0	1	0	0	−1	0	0	1	0
L_{14}	1	1	0	0	1	0	0	−1	1	0	0
L_{23}	1	0	1	1	0	0	0	1	−1	0	0
L_{24}	1	0	1	0	1	0	1	0	0	−1	0
L_{34}	1	0	0	1	1	1	0	0	0	0	−1

Table 11.1: Intersections on S - del Pezzo 5

respectively, and L_{ij} be the preimages of lines connecting b_i and b_j for $i \in \{1,...,4\}$. Recall that E_1, E_2, E_3, E_4 and L_{ij} generate the Mori Cone $\overline{\mathrm{NE}(S)}$. We have that

$$-K_X \sim \pi^*(4H) - E_C - E_L,$$

$$R \sim \pi^*(4H) - 2E_C - E_L,$$

and moreover

$$\pi^*(H)|_S \sim S|_S \sim Z, \quad E_L|_S \sim E_1, \quad E_C|_S \sim E_2 + E_3 + E_4.$$

The intersections are given by:

By Theorem 11.2.4, we have $S_X(S) < 1$. By Corollary 11.2.5 we have $S(W^S_{\bullet,\bullet};Z) \geqslant 1$. Let us compute $S(W^S_{\bullet,\bullet};Z)$. Take $u \in \mathbb{R}_{\geqslant 0}$. Observe that

$$-K_X - uS \sim_{\mathbb{R}} (1-u/3)R + u/3(\pi^*(H) - E_L) + (1-2u/3)E_C,$$

which implies that $-K_X - uS$ is pseudo-effective if and only if $u \leqslant \frac{3}{2}$ since $\zeta_*(-K_X - uS)$ is a divisor of degree $(u/3, 1-2u/3)$ on $\mathbb{P}^1 \times \mathbb{P}^2$, which implies that $-K_X - uS$ is not pseudo-effective for $u > 3/2$. Let $P(u) = P(-K_X - uS)$ be a positive part of Zariski decomposition and $N(u) = N(-K_X - uS)$ be a negative part of Zariski decomposition. Here we use the notations introduced in Theorem 11.2.5.

$$P(u) = \begin{cases} -K_X - uS \text{ if } 0 \leqslant u \leqslant 1, \\ -K_X - uS - (u-1)R \text{ if } 1 \leqslant u \leqslant \dfrac{3}{2}. \end{cases} \text{ and } N(u) = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ (u-1)R \text{ if } 1 \leqslant u \leqslant \dfrac{3}{2}. \end{cases}$$

Then take any $v \in \mathbb{R}_{\geqslant 0}$. Suppose $P(u,v)$ is a positive part of the Zariski decomposition of $(-K_X - uS)|_S - vZ$ and $N(u,v)$ is a negative part of the Zariski decomposition of $(-K_X - uS)|_S - vZ$. The intersections of $P(u,v)$ with the generators of $\overline{\mathrm{NE(S)}}$ above are:

	$P(u,v)$				
u	$[0,1]$	$[1,7/5]$		$[7/5,3/2]$	
v	$[0,2-u]$	$[0,3-2u]$	$\left[3-2u,\frac{5-3u}{2}\right]$	$[0,3-2u]$	$[3-2u,6-4u]$
E_1	1	$2-u$	$11-7u-3v$	$2-u$	$11-7u-3v$
E_2, E_3, E_4	1	$3-2u$	$6-4u-v$	$3-2u$	$6-4u-v$
L_{12}, L_{13}, L_{14}	$2-u-v$	$3-2u-v$	0	$3-2u-v$	0
L_{23}, L_{24}, L_{34}	$2-u-v$	$2-u-v$	$5-3u-2v$	$2-u-v$	$5-3u-2v$

Table 11.2: Zariski Decomposition of $(-K_X-uS)|_S-vZ$ for S - del Pezzo 5

So we obtain:

- if $u\in[0,1]$ and $v\in[0,2-u]$ then:

$$P(u,v)=(4-u-v)Z-(E_1+E_2+E_3+E_4)\ \text{ and }\ N(u,v)=0.$$

- if $u\in[1,3/2]$ and $v\in[0,3-2u]$ then

$$P(u,v)=(8-5u-v)Z+(2u-3)(E_2+E_3+E_4)+(u-2)E_1\ \text{ and }\ N(u,v)=0.$$

- if $u\in[1,7/5]$ and $v\in\left[3-2u,\frac{5-3u}{2}\right]$ or $u\in[7/5,3/2]$ and $v\in[3-2u,6-4u]$ then:

$$P(u,v)=(8-5u-v)Z+(2u-3)(E_2+E_3+E_4)+(u-2)E_1+(3-2u-v)(L_{12}+L_{13}+L_{14}),$$

$$N(u,v)=(2u+v-3)(L_{12}+L_{13}+L_{14}).$$

Corollary 11.2.5 gives us

$$S\big(W^S_{\bullet,\bullet};Z\big)=\frac{3}{(-K_X)^3}\int_0^{3/2}\big(P(u)^2\cdot S\big)\cdot\mathrm{ord}_Z\Big(N(u)\big|_S\Big)du+\frac{3}{(-K_X)^3}\int_0^{3/2}\int_0^{\infty}\mathrm{vol}\big(P(u)\big|_S-vZ\big)dvdu.$$

Note that $\mathrm{ord}_Z\Big(N(u)\big|_S\Big)=0$ because $Z\not\subset R$. So we are only left with the second part of the integral which equals:

$$\begin{aligned}S\big(W^S_{\bullet,\bullet};Z\big)=&\frac{3}{28}\int_0^1\int_0^{2-u}\big((4-u-v)Z-(E_1+E_2+E_3+E_4)\big)^2dvdu+\\&+\frac{3}{28}\int_1^{3/2}\int_0^{3-2u}\big((8-5u-v)Z+(2u-3)(E_2+E_3+E_4)+(u-2)E_1\big)^2dvdu+\\&+\frac{3}{28}\int_1^{7/5}\int_{3-2u}^{\frac{5-3u}{2}}\big((8-5u-v)Z+(2u-3)(E_2+E_3+E_4)+(u-2)E_1)+\\&\qquad+(3-2u-v)(L_{12}+L_{13}+L_{14})\big)^2dvdu+\\&+\frac{3}{28}\int_{7/5}^{3/2}\int_{3-2u}^{6-4u}\big((8-5u-v)Z+(2u-3)(E_2+E_3+E_4)+(u-2)E_1)+\\&\qquad+(3-2u-v)(L_{12}+L_{13}+L_{14})\big)^2dvdu=\end{aligned}$$

$$=\frac{3}{28}\int_0^1\int_0^{2-u}\big((4-u-v)^2-4\big)dvdu+$$
$$+\frac{3}{28}\int_1^{3/2}\int_0^{2u-3}\big(12u^2+10uv+v^2-40u-16v+33\big)dvdu+$$
$$+\frac{3}{28}\int_1^{7/5}\int_{3-2u}^{\frac{5-3u}{2}}\big(24u^2+22uv+4v^2-76u-34v+60\big)dvdu+$$
$$+\frac{3}{28}\int_1^{7/5}\int_{3-2u}^{6-4u}\big(24u^2+22uv+4v^2-76u-34v+60\big)dvdu=\frac{753}{1120}<1.$$

The obtained contradiction completes the proof of the lemma. □

Lemma 11.4.2. *One has $Z \not\subset E_L$.*

Proof. Suppose that $Z \subset E_L$. Observe that $E_L \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathbf{s}$ be the section of the natural projection $E_L \to L$ such that $\mathbf{s}^2 = 0$, and $\mathbf{l}$ be a fiber of this projection. The intersections are given by: Then

	$\mathbf{s}$	$\mathbf{l}$
$\mathbf{s}$	0	1
$\mathbf{l}$	1	0

Table 11.3: Intersections on $E_L \cong \mathbb{P}^1 \times \mathbb{P}^1$

$$E_L|_{E_L} \sim -\mathbf{s}+\mathbf{l},\ \pi^*(H)|_{E_L} \sim \mathbf{l},\ R|_{E_L} \sim \mathbf{s}+3\mathbf{l},$$

By Theorem 11.2.4, we have $S_X(E_L) < 1$. Thus, we conclude that $S(W^{E_L}_{\bullet,\bullet};Z) \geqslant 1$ by Corollary 11.2.5. Let us compute $S(W^{E_L}_{\bullet,\bullet};Z)$. Take $u \in \mathbb{R}_{\geqslant 0}$. Observe that

$$-K_X - uE_L \sim_{\mathbb{R}} \frac{1}{2}R + 2(\pi^*(H) - E_L) + \Big(\frac{3}{2} - u\Big)E_L,$$

which implies that $-K_X - uE_L$ is pseudo-effective if and only if $u \leqslant \frac{3}{2}$ since $\zeta_*(-K_X - uS)$ is a divisor of degree $(2, 3/2 - u)$ on $\mathbb{P}^1 \times \mathbb{P}^2$, which implies that $-K_X - uS$ is not pseudo-effective for $u > 3/2$. Let $P(u) = P(-K_X - uE_L)$ and $N(u) = N(-K_X - uE_L)$. Then

$$P(u) = \begin{cases} -K_X - uE_L \text{ for } 0 \leqslant u \leqslant 1, \\ (8-4u)\pi^*(H) - (3-2u)E_C - 2E_L \text{ for } 1 \leqslant u \leqslant \dfrac{3}{2}. \end{cases}$$

and

$$N(u) = \begin{cases} 0 \text{ for } 0 \leqslant u \leqslant 1, \\ (u-1)R \text{ for } 1 \leqslant u \leqslant \dfrac{3}{2}. \end{cases}$$

Then take any $v \in \mathbb{R}_{\geqslant 0}$. Suppose $P(u,v)$ is a positive part of the Zariski decomposition of $(-K_X - uE_L)|_{E_L} - vZ$, $N(u,v)$ is a negative part of the Zariski decomposition of $(-K_X - uE_L)|_{E_L} - vZ$.

$\boxed{1}$ Suppose that $Z \neq R|_{E_L}$ and $Z \sim a\mathbf{s} + b\mathbf{l}$. Note that $a \geqslant 1$ since $\mathbb{P}^3$ does not contain G-fixed points. Then using the convexity of volume we get the inequality we obtain

$$S\big(W^{E_L}_{\bullet,\bullet};Z\big) = \frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty} \mathrm{vol}\Big(P(u)\big|_{E_L} - v(a\mathbf{s}+b\mathbf{l})\Big)dvdu \leqslant \frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty} \mathrm{vol}\Big(P(u)\big|_{E_L} - v\mathbf{s}\Big)dvdu.$$

so it is enough to show that the last integral is less than 1 to deduce a contradiction. So suppose $Z \sim \mathbf{s}$. We have that:

$$P(u)|_{E_L} - v\mathbf{s} \sim_{\mathbb{R}} \begin{cases} (1+u-v)\mathbf{s} + (3-u)\mathbf{l} \text{ for } 0 \leqslant u \leqslant 1, \\ (2-v)\mathbf{s} + (6-4u)\mathbf{l} \text{ for } 1 \leqslant u \leqslant 3/2. \end{cases}$$

The intersections of $P(u,v)$ with the generators of $\overline{\mathrm{NE}(S)}$ are: Then Corollary 11.2.5 gives

	$P(u,v)$	
u	$[0,1]$	$[1,3/2]$
v	$[0,1+u]$	$[0,2]$
$\mathbf{s}$	$3-u$	$6-4u$
$\mathbf{l}$	$1+u-v$	$2-v$

Table 11.4: Zariski Decomposition of $(-K_X - uE_L)|_{E_L} - vZ$ for $E_L \cong \mathbb{P}^1 \times \mathbb{P}^1$ (case 1)

$$S\big(W^{E_L}_{\bullet,\bullet};\mathbf{s}\big) = \frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty} \mathrm{vol}\Big(P(u)\big|_{E_L} - v\mathbf{s}\Big)dvdu =$$

$$\frac{3}{28}\int_0^1\int_0^{1+u} 2(3-u)(1+u-v)dvdu + \frac{3}{28}\int_1^{3/2}\int_0^2 4(v-2)(-3+2u)dvdu = \frac{13}{16} < 1.$$

Hence, for any G-invariant curve $Z \subset E_L$ such that $Z \neq R|_{E_L}$ we have $S\big(W^{E_L}_{\bullet,\bullet};Z\big) < 1$ so by Corollary 11.2.5 we obtain the desired contradiction.

$\boxed{2}$ Suppose $Z = R|_{E_L} \sim \mathbf{s} + 3\mathbf{l}$. Take any $v \in \mathbb{R}_{\geqslant 0}$ then we have:

$$P(u)|_{E_L} - vZ \sim_{\mathbb{R}} \begin{cases} (1+u-v)\mathbf{s} + (3-u-3v)\mathbf{l}, \text{ for } 0 \leqslant u \leqslant 1, \\ (2-v)\mathbf{s} + (6-4u-3v)\mathbf{l}, \text{ for } 1 \leqslant u \leqslant 3/2. \end{cases}$$

The intersections of $P(u,v)$ with the generators of $\overline{\mathrm{NE}(S)}$ are: Hence, if $Z = R|_{E_L}$, then Corollary 11.2.5 gives

	$P(u,v)$	
u	$[0,1]$	$[1,3/2]$
v	$[0,\frac{3-u}{3}]$	$[0,\frac{6-4u}{3}]$
$\mathbf{s}$	$3-u-3v$	$6-4u-3v$
$\mathbf{l}$	$1+u-v$	$2-v$

Table 11.5: Zariski Decomposition of $(-K_X-uE_L)|_{E_L}-vZ$ for $E_L\cong\mathbb{P}^1\times\mathbb{P}^1$ (case 2)

$$S\big(W^{E_L}_{\bullet,\bullet};Z\big)=\frac{3}{28}\int_1^{\frac{3}{2}}(u-1)E_L\cdot\big((8-4u)\pi^*(H)-(3-2u)E_C-2E_L\big)^2du+$$
$$+\frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty}\mathrm{vol}\Big(P(u)\big|_{E_L}-vZ\Big)dvdu=$$
$$=\frac{3}{28}\int_1^{\frac{3}{2}}4(u-1)(6-4u)du+\frac{3}{28}\int_0^1\int_0^{\frac{3-u}{3}}2(1+u-v)(3-u-3v)dvdu+$$
$$+\frac{3}{28}\int_1^{\frac{3}{2}}\int_0^{\frac{6-4u}{3}}2(2-v)(6-4u-3v)dvdu=\frac{19}{56}<1.$$

Hence, by Corollary 11.2.5 we obtain the desired contradiction which completes the proof of the lemma. □

Lemma 11.4.3. *Let $\mathscr{S}$ be a G-invariant surface in $|\pi^*(2H)-E_C|$. Then $Z\not\subset\mathscr{S}$.*

Proof. Suppose that $Z\subset\mathscr{S}$. Note that $\pi(\mathscr{S})$ is a G-invariant quadric that contains C. Recall that it is smooth. Let's seek for a contradiction. Let us identify $\pi(\mathscr{S})=\mathbb{P}^1\times\mathbb{P}^1$ such that C is a curve in $\pi(\mathscr{S})$ of degree $(1,2)$. Then π induces a birational morphism $\varphi\colon\mathscr{S}\to\mathbb{P}^1\times\mathbb{P}^1$ that is a blow up of two intersection points $\pi(\mathscr{S})\cap L=\{A_1,A_2\}$, which are not contained in the curve C. Moreover, the surface $\mathscr{S}$ is a smooth del Pezzo surface of degree 6, because the points of the intersection $\pi(\mathscr{S})\cap L$ are not contained in one line in $\pi(\mathscr{S})$ since otherwise this line would be L. But L is not contained in $\pi(\mathscr{S})$ which is a contradiction. By Theorem 11.2.4, we have $S_X(\mathscr{S})<1$. Let's show that $S(W^{\mathscr{S}}_{\bullet,\bullet};Z)<1$.
Take $u\in\mathbb{R}_{\geqslant0}$. Then

$$-K_X-u\mathscr{S}\sim_{\mathbb{R}}2\pi^*(H)-E_L+(1-u)(2\pi^*(H)-E_C),$$

which implies that $-K_X-u\mathscr{S}$ is nef for every $u\in[0,1]$. On the other hand, we have

$$-K_X-u\mathscr{S}\sim_{\mathbb{R}}(4-2u)\big(\pi^*(H)-E_L\big)+(3-2u)E_L+(u-1)E_C,$$

so that the divisor $-K_X-uS$ is pseudo-effective $\iff u\in\big[0,\frac{3}{2}\big]$. We denote $P(u)=P(-K_X-u\mathscr{S})$ and $N(u)=N(-K_X-u\mathscr{S})$. Then we have

$$P\big(u\big)=\begin{cases}-K_X-u\mathscr{S}\text{ for }0\leqslant u\leqslant1,\\(4-2u)\pi^*(H)-E_L\text{ for }1\leqslant u\leqslant\dfrac{3}{2}.\end{cases}\text{ and }N(u)=\begin{cases}0\text{ for }0\leqslant u\leqslant1,\\(u-1)E_C\text{ for }1\leqslant u\leqslant\dfrac{3}{2}.\end{cases}$$

Suppose $\varphi : \mathscr{S} \to \mathbb{P}^1 \times \mathbb{P}^1$ is the blowup at points A_1, A_2 with e_1 and e_2 the exceptional curves of φ which correspond to points A_1, A_2 respectively. We denote by ℓ_1 and ℓ_2 the proper transforms on $\mathscr{S}$ of general curves k_1 and k_2 in $\mathbb{P}^1 \times \mathbb{P}^1$ of degrees $(1,0)$ and $(0,1)$, respectively. The intersections are given by: Then

	ℓ_1	ℓ_2	e_1	e_2
ℓ_1	0	1	0	0
ℓ_2	1	0	0	0
e_1	0	0	-1	0
e_1	0	0	0	-1

Table 11.6: Intersections on $\mathscr{S}$ - del Pezzo 6 (pic. 1)

$$\pi^*(H)|_{\mathscr{S}} \sim \ell_1 + \ell_2,\ E_L|_{\mathscr{S}} \sim e_1 + e_2,\ E_C|_{\mathscr{S}} \sim \ell_1 + 2\ell_2.$$

We let F_{11}, F_{12}, F_{21}, F_{22} be the (-1)-curves on $\mathscr{S}$ such that

$$F_{11} \sim \ell_1 - e_1,\ F_{12} \sim \ell_1 - e_2,\ F_{21} \sim \ell_2 - e_1,\ F_{22} \sim \ell_2 - e_2.$$

We have $\overline{\mathrm{NE}(\mathscr{S})} =< e_1, e_2, F_{11}, F_{12}, F_{21}, F_{22} >$ with the intersections:

$\boxed{1}$ Suppose $Z \neq E_C|_{\mathscr{S}}$, then $\varphi(Z)$ is a curve since since $Z \neq e_1$ and $Z \neq e_2$, because neither e_1 nor e_2 is G-invariant. Now we have that $\varphi(Z) \sim ak_1 + bk_2$ and so

$$Z \sim a\ell_1 + b\ell_2 - m_1 e_1 - m_2 e_2,$$

where m_1 is the multiplicity of $\varphi(Z)$ at point A_1, m_2 is the multiplicity of $\varphi(Z)$ at point A_2. Note that G exchanges A_1 and A_2 and Z is a G-invariant curve thus $m_1 = m_2 =: m$. We know that $Z \notin \{F_{11}, F_{12}, F_{21}, F_{22}\}$ since the F_{ij} are not G-invariant for any i, j. Thus:

$$0 \leqslant Z \cdot F_{11} = (a\ell_1 + b\ell_2 - m_1 e_1 - m_2 e_2)(\ell_1 - e_1) = b - m \Rightarrow b \geqslant m,$$

$$0 \leqslant Z \cdot F_{22} = (a\ell_1 + b\ell_2 - m_1 e_1 - m_2 e_2)(\ell_2 - e_2) = a - m \Rightarrow a \geqslant m.$$

	e_1	e_2	F_{11}	F_{12}	F_{21}	F_{22}
e_1	-1	0	1	0	1	0
e_2	0	-1	0	1	0	1
F_{11}	1	0	-1	0	0	1
F_{12}	0	1	0	-1	1	0
F_{21}	1	0	0	1	-1	0
F_{22}	0	1	1	0	0	-1

Table 11.7: Intersections on $\mathscr{S}$ - del Pezzo 6 (pic. 2)

Now we have that

$$Z \sim a\ell_1 + b\ell_2 - m(e_1+e_2) =$$

$$= \begin{cases} \underbrace{(a-b)}_{\geqslant 0}\ell_1 + \underbrace{m}_{\geqslant 0}(\ell_1+\ell_2-e_1-e_2) + \underbrace{(b-m)}_{\geqslant 0}(\ell_1+\ell_2), \text{ for } a \geqslant b, \\ \underbrace{(b-a)}_{\geqslant 0}\ell_2 + \underbrace{m}_{\geqslant 0}(\ell_1+\ell_2-e_1-e_2) + \underbrace{(a-m)}_{\geqslant 0}(\ell_1+\ell_2), \text{ for } b \geqslant a. \end{cases}$$

So we can decompose each curve Z as the sum of ℓ_1, ℓ_2, $\ell_1+\ell_2-e_1-e_2$ with non-negative coefficients, i.e. $Z \sim c_1\ell_1 + c_2\ell_2 + c_3(\ell_1+\ell_2-e_1-e_2)$ for some c_1, c_2, $c_3 \in \mathbb{Z}_{\geqslant 0}$. Note if for example $c_1 \geqslant 1$ then using the convexity of volume we get the inequality:

$$S\big(W^{\mathscr{S}}_{\bullet,\bullet};Z\big) = \frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty} \mathrm{vol}\Big(P(u)\big|_{\mathscr{S}} - v(c_1\ell_1+c_2\ell_2+c_3(\ell_1+\ell_2-e_1-e_2))\Big)dvdu \leqslant$$

$$\leqslant \frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty} \mathrm{vol}\Big(P(u)\big|_{\mathscr{S}} - v\ell_1\Big)dvdu.$$

and similarly for c_2 and c_3. So it is enough to get $S\big(W^{\mathscr{S}}_{\bullet,\bullet};Z\big) < 1$ for $Z \sim \ell_1$, $Z \sim \ell_2$, $Z \sim \ell_1+\ell_2-e_1-e_2$ to deduce a contradiction. Take any $v \in \mathbb{R}_{\geqslant 0}$. Suppose $P(u,v)$ and $N(u,v)$ are positive and negative parts part of the Zariski decomposition of $(-K_X - u\mathscr{S})|_{\mathscr{S}} - vZ$ respectively.

Case 1. Suppose $Z \sim \ell_1$ then the intersections of $P(u,v)$ with the generators of $\overline{\mathrm{NE}(\mathscr{S})}$ above are given in the following table: So we obtain:

	$P(u,v)$			
u	$[0,1]$		$[1,3/2]$	
v	$[0,2-u]$	$[2-u,3-u]$	$[0,3-2u]$	$[3-2u,6-4u]$
e_1, e_2	1	$3-u-v$	1	$4-2u-v$
F_{11}, F_{12}	1	$3-u-v$	$3-2u$	$6-4u-v$
F_{21}, F_{22}	$2-u-v$	0	$3-2u-v$	0

Table 11.8: Zariski Decomposition of $(-K_X - u\mathscr{S})|_{\mathscr{S}} - vZ$ for $\mathscr{S}$ - del Pezzo 6 (case 1)

- if $u \in [0,1]$ then:

$$P(u,v) = \begin{cases} (3-u-v)\ell_1 + 2\ell_2 - (e_1+e_2) \text{ for } 0 \leqslant v \leqslant 2-u, \\ (3-u-v)(\ell_1+\ell_2-e_1-e_2) \text{ for } 2-u \leqslant v \leqslant 3-u. \end{cases}$$

and

$$N(u,v) = \begin{cases} 0 \text{ for } 0 \leqslant v \leqslant 2-u, \\ (u+v-2)(2\ell_2-e_1-e_2) \text{ for } 2-u \leqslant v \leqslant 3-u. \end{cases}$$

- if $u \in [1,3/2]$ then:

$$P(u,v) = \begin{cases} (4-2u-v)\ell_1 + (4-2u)\ell_2 - (e_1+e_2) \text{ for } 0 \leqslant v \leqslant 3-2u, \\ (4-2u-v)(\ell_1 - e_1 - e_2) + (10-6u-2v)\ell_2 \text{ for } 3-2u \leqslant v \leqslant 6-4u. \end{cases}$$

and

$$N(u,v) = \begin{cases} 0 \text{ for } 0 \leqslant v \leqslant 3-2u, \\ (3-2u-v)(2\ell_2 - e_1 - e_2) \text{ for } 3-2u \leqslant v \leqslant 6-4u. \end{cases}$$

Corollary 11.2.5 gives us:

$$\begin{aligned} S(W^{\mathscr{S}}_{\bullet,\bullet};Z) &= \frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty} \mathrm{vol}\Big(P(u)\big|_{\mathscr{S}} - v\ell_1\Big)dvdu = \\ &= \frac{3}{28}\int_0^1\int_0^{2-u}\Big(10-4u-4v\Big)dvdv + \frac{3}{28}\int_0^1\int_{2-u}^{3-u} 2(3-u-v)^2 dvdu + \\ &\quad + \frac{3}{28}\int_1^{\frac{3}{2}}\int_0^{3-2u}\Big(8u^2+4uv-32u-8v+30\Big)dvdu + \\ &\quad + \frac{3}{28}\int_1^{\frac{3}{2}}\int_{3-2u}^{6-4u} 2(4-2u-v)(6-4u-v)dvdu = \frac{109}{112}. \end{aligned}$$

Hence, by Corollary 11.2.5 we obtain the desired contradiction.

Case 2. Suppose $Z \sim \ell_2$ then the intersections of $P(u,v)$ with the generators of $\overline{\mathrm{NE}(\mathscr{S})}$ are given in the following table: So, we obtain:

	$P(u,v)$			
u	$[0,1]$		$[1,3/2]$	
v	$[0,1]$	$[1,2]$	$[0,3-2u]$	$[3-2u,6-4u]$
e_1, e_2	1	$2-v$	1	$4-2u-v$
F_{11}, F_{12}	$1-v$	0	$3-2u-v$	0
F_{21}, F_{22}	$2-u$	$3-u-v$	$3-2u$	$6-4u-v$

Table 11.9: Zariski Decomposition of $(-K_X - u\mathscr{S})|_{\mathscr{S}} - vZ$ for $\mathscr{S}$ - del Pezzo 6 (case 2)

- if $u \in [0,1]$ then:

$$P(u,v) = \begin{cases} (3-u)\ell_1 + (2-v)\ell_2 - (e_1+e_2) \text{ for } 0 \leqslant v \leqslant 1, \\ (v-2)(e_1+e_2) + (5-u-2v)\ell_1 + (2-v)\ell_2 \text{ for } 1 \leqslant v \leqslant 2. \end{cases}$$

and

$$N(u,v) = \begin{cases} 0 \text{ for } 0 \leqslant v \leqslant 1, \\ (1-v)(2\ell_1 - e_1 - e_2) \text{ for } 1 \leqslant v \leqslant 2. \end{cases}$$

- if $u \in [1,3/2]$ then:

$$P(u,v) = \begin{cases} (4-2u)\ell_1 + (4-2u-v)\ell_2 - (e_1+e_2) \text{ for } 0 \leqslant v \leqslant 3-2u, \\ (4-2u-v)(\ell_2 - e_1 - e_2) + (10-6u-2v)\ell_1 \text{ for } 3-2u \leqslant v \leqslant 6-4u. \end{cases}$$

and

$$N(u,v) = \begin{cases} 0 \text{ for } 0 \leqslant v \leqslant 3-2u, \\ (3-2u-v)(2\ell_1 - e_1 - e_2) \text{ for } 3-2u \leqslant v \leqslant 6-4u. \end{cases}$$

Corollary 11.2.5 gives us:

$$\begin{aligned} S\big(W^{\mathscr{S}}_{\bullet,\bullet};Z\big) &= \frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty} \mathrm{vol}\Big(P(u)\big|_{\mathscr{S}} - v\ell_2\Big)dvdu = \\ &= \frac{3}{28}\int_0^1\int_0^1 \Big(2uv - 4u - 6v + 10\Big)dvdv + \frac{3}{28}\int_0^1\int_1^2 2(v-2)(v-3+u)dvdu + \\ &\quad + \frac{3}{28}\int_1^{\frac{3}{2}}\int_0^{3-2u}\Big(8u^2 + 4uv - 32u - 8v + 30\Big)dvdu + \\ &\quad + \frac{3}{28}\int_1^{\frac{3}{2}}\int_{3-2u}^{6-4u} 2(4-2u-v)(6-4u-v)dvdu = \frac{89}{112} < 1. \end{aligned}$$

Hence, by Corollary 11.2.5 we obtain the desired contradiction.

Case 3. Suppose $Z \sim \ell_1 + \ell_2 - e_1 - e_2$ then the intersections of $P(u,v)$ the generators of $\overline{\mathrm{NE}}(\mathscr{S})$ are given in the following table: So we obtain:

	$P(u,v)$			
u	$[0,1]$		$[1,3/2]$	
v	$[0,1]$	$[1,2]$	$[0,1]$	$[1,4-2u]$
e_1, e_2	$1-v$	0	$1-v$	0
F_{11}, F_{12}	1	$2-v$	$3-2u$	$4-2u-v$
F_{21}, F_{22}	$2-u$	$3-u-v$	$3-2u$	$4-2u-v$

Table 11.10: Zariski Decomposition of $(-K_X - u\mathscr{S})|_{\mathscr{S}} - vZ$ for $\mathscr{S}$ - del Pezzo 6 (case 3)

- if $u \in [0,1]$ then:

$$P(u,v) = \begin{cases} (v-1)(e_1+e_2) + (3-u-v)\ell_1 + (2-v)\ell_2 \text{ for } 0 \leqslant v \leqslant 1, \\ (3-u-v)\ell_1 + (2-v)\ell_2 \text{ for } 1 \leqslant v \leqslant 2. \end{cases}$$

and

$$N(u,v) = \begin{cases} 0 \text{ for } 0 \leqslant v \leqslant 1, \\ (1-v)(e_1+e_2) \text{ for } 1 \leqslant v \leqslant 2. \end{cases}$$

- if $u \in [1,3/2]$ then:

$$P(u,v) = \begin{cases} (v-1)(e_1+e_2)+(4-2u-v)(\ell_1+\ell_2) \text{ for } 0 \leqslant v \leqslant 1, \\ (4-2u-v)(\ell_1+\ell_2) \text{ for } 1 \leqslant v \leqslant 4-2u. \end{cases}$$

and

$$N(u,v) = \begin{cases} 0 \text{ for } 0 \leqslant v \leqslant 1, \\ (1-v)(e_1+e_2) \text{ for } 1 \leqslant v \leqslant 4-2u. \end{cases}$$

Corollary 11.2.5 gives us:

$$\begin{aligned} S\big(W^{\mathscr{S}}_{\bullet,\bullet};Z\big) &= \frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty}\mathrm{vol}\Big(P(u)\big|_{\mathscr{S}}-vZ\Big)dvdu = \\ &= \frac{3}{28}\int_0^1\int_0^1\Big(2uv-4u-6v+10\Big)dvdu+\frac{3}{28}\int_0^1\int_1^2 2(v-2)(u+v-3)dvdu+ \\ &+\frac{3}{28}\int_1^{\frac{3}{2}}\int_0^1\Big(2(2u-3)(2u+2v-5)\Big)dvdu+\frac{3}{28}\int_1^{\frac{3}{2}}\int_1^{4-2u}2(2u+v-4)^2dvdu = \\ &= \frac{13}{16}<1. \end{aligned}$$

Hence, by Corollary 11.2.5 we obtain the desired contradiction. Thus, for any G-invariant curve $Z \subset \mathscr{S}$ such that $Z \neq E_C|_{\mathscr{S}}$ we have $S\big(W^{\mathscr{S}}_{\bullet,\bullet};Z\big) < 1$ which is impossible by Corollary 11.2.5.

$\boxed{2}$ Suppose $Z = E_C|_{\mathscr{S}} \sim \ell_1+2\ell_2$ then using the convexity of volume we get the inequality:

$$\begin{aligned} S\big(W^{\mathscr{S}}_{\bullet,\bullet};Z\big) &= \frac{3}{28}\int_0^{\frac{3}{2}}\Big(P(u)\cdot P(u)\cdot\mathscr{S}\Big)\mathrm{ord}_Z\Big(N(u)\big|_{\mathscr{S}}\Big)du+\frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty}\mathrm{vol}\Big(P(u)\big|_{\mathscr{S}}-vZ\Big)dvdu = \\ &= \frac{3}{28}\int_1^{\frac{3}{2}}(u-1)\big((4-2u)\pi^*(H)-E_L\big)^2\cdot\big(2\pi^*(H)-E_C\big)du+\frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty}\mathrm{vol}\Big(P(u)\big|_{\mathscr{S}}-vZ\Big)dvdu = \\ &= \frac{3}{28}\int_1^{\frac{3}{2}}(u-1)\big(2(4-2u)^2-2\big)du+\frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty}\mathrm{vol}\Big(P(u)\big|_{\mathscr{S}}-vZ\Big)dvdu = \\ &= \frac{5}{224}+\frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty}\mathrm{vol}\Big(P(u)\big|_{\mathscr{S}}-v(\ell_1+2\ell_2)\Big)dvdu \leqslant \\ &\leqslant \frac{5}{224}+\frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty}\mathrm{vol}\Big(P(u)\big|_{\mathscr{S}}-v\ell_1\Big)dvdu = \\ &= \frac{5}{224}+\frac{109}{112}=\frac{223}{224}<1. \end{aligned}$$

Hence, by Corollary 11.2.5 we obtain the desired contradiction which completes the proof of the lemma. □

Corollary 11.4.4. *The curve $\pi(Z)$ is not a line that intersects C.*

Proof. Recall from Section 11.3.3 that there are exactly 3 G-invariant lines in $\mathbb{P}^3$ that intersect the curve C. These are the lines L_{12}, L_{34}, L_{56}. We have that $L_{12} \subset Q_2 \cap Q_3$, $L_{34} \subset Q_1 \cap Q_2$, $L_{34} \subset Q_1 \cap Q_3$. Thus, the lemma above gives us the result. □

By Lemma 11.2.7, one has $\alpha_{G,Z}(X) < \frac{3}{4}$. Thus, by lemma (Araujo et al., 2023, Lemma 1.42) and its proof, there is a G-invariant effective $\mathbb{Q}$-divisor D on the threefold X such that $D \sim_{\mathbb{Q}} -K_X$ and $Z \subseteq \mathrm{Nklt}(X, \lambda D)$ for some positive rational number $\lambda < \frac{3}{4}$.

Lemma 11.4.5. *Let S be an irreducible surface in X. Suppose that $S \subset \mathrm{Nklt}(X, \lambda D)$. Then either $S \in |\pi^*(2H) - E_C|$ and S is G-invariant or $S = E_L$.*

Proof. Consider a G-irreducible surface T such that S is its component. Note that $T \in \mathrm{Nklt}(X, \lambda D)$ and is G-invariant. We have $D \sim_{\mathbb{Q}} 4\pi^*(H) - E_C - E_L$ and $\lambda < \frac{3}{4}$. By assumption we have $D = aT + \Delta$ where $a \in \mathbb{Q}$ such that $a \geqslant \frac{1}{\lambda} > \frac{4}{3}$ and Δ is an effective $\mathbb{Q}$ divisor on X whose support does not contain T. If $T = E_C$ we get

$$\pi^*(4H) - E_C - E_L \sim_{\mathbb{Q}} aE_C + \Delta \Rightarrow \Delta \sim_{\mathbb{Q}} \pi^*(4H) - (1+a)E_C - E_L = R - (a-1)E_C,$$

which gives a contradiction.
Assume $T \neq E_L$, $T \neq E_C$ then $\pi(T) \subset \mathbb{P}^3$ is the surface of some degree d. We have that:

$$4H \sim_{\mathbb{Q}} a\pi(T) + \pi(\Delta) \Rightarrow 4 \geqslant ad \Rightarrow d = 1 \text{ or } d = 2.$$

The latter holds since $a > \frac{4}{3}$. Since $\mathbb{P}^3$ does not contain G-invariant points, it contains no G-invariant lines, so $\pi(T)$ is a quadric. Then T is given by

$$T \sim 2\pi^*(H) - m_L E_L - m_C E_C,$$

where m_L and m_C are multiplicities of $\pi(T)$ at general points of the curves L and C respectively. By Corollary 11.3.7 the cone of effective divisors over $\mathbb{Q}$ is generated by E_L, E_C, $R \sim 4\pi^*(H) - 2E_C - E_L$ and $\pi^*(H) - E_L$ so for some $a_1, a_2, a_3, a_4 \in \mathbb{Q}_{\geqslant 0}$ we have

$$\Delta \sim D - aT \sim a_1 E_L + a_2 E_C + a_3(4\pi^*(H) - 2E_C - E_L) + a_4(\pi^*(H) - E_L).$$

It follows that

$$(4-2a)\pi^*(H) + (am_L - 1)E_L + (am_C - 1)E_C \sim$$
$$\sim a_1 E_L + a_2 E_C + a_3(4\pi^*(H) - 2E_C - E_L) + a_4(\pi^*(H) - E_L).$$

By comparing coefficients we get a linear system

$$\begin{cases} 4 - 2a = 4a_3 + a_4, \\ a_3 + a_4 - a_1 = 1 - am_L, \\ 2a_3 - a_2 = 1 - am_C, \\ a_1 \geqslant 0,\ a_2 \geqslant 0,\ a_3 \geqslant 0,\ a_4 \geqslant 0,\ a > 4/3. \end{cases}$$

If T is reducible (union of 2 planes), then $m_C = 0$, $m_L \in \{0,1,2\}$. Suppose $m_C = 0$ then this system above gives us a contradiction:

$$2 > 4 - 2a = 4a_3 + a_4 \geqslant 4a_3 = 2(1 + a_2) \geqslant 2.$$

Thus, T is irreducible. Then $m_L = 0$ and $m_C = 1$, because L and C are disjoint. We see that the only options are $S \in |\pi^*(2H) - E_C|$ and S is G-invariant or $S = E_L$. □

Corollary 11.4.6. *One has $Z \not\subset E_C$.*

Proof. Suppose that $Z \subset E_C$. Observe that $\pi(Z)$ is not a point, since $\mathbb{P}^3$ does not have G-fixed points by Lemma 11.3.1. Hence, we see that $\pi(Z)$ is the twisted cubic C.
Let S be a general fiber of η. Then $S \cdot Z \geqslant 3$, which contradicts Lemma 11.2.9. □

Lemma 11.4.7. *The curve $\pi(Z)$ is a line.*

Proof. Let $\overline{D} = \pi(D)$, $\overline{Z} = \pi(Z)$. We see that $\overline{Z}$ is a G-invariant curve in $\mathbb{P}^3$ such that such that Z is not contained in a G-invariant surface in $|2\pi^*(H) - E_C|$, $Z \not\subset E_L$, and $Z \not\subset E_C$ (by Lemmas 11.4.3 , 11.4.2, and 11.4.6 respectively). Then $\overline{Z} \subset (\mathbb{P}^3, \lambda\overline{D})$ and $\overline{Z}$ is not contained in any surface contained in $\mathrm{Nklt}(\mathbb{P}^3, \lambda\overline{D})$ by Lemma 11.4.5. Now we apply Lemma 11.2.8 and get that $\mathscr{O}_{\mathbb{P}^3}(1) \cdot \overline{Z} \leqslant 1$. Thus $\mathscr{O}_{\mathbb{P}^3}(1) \cdot \overline{Z} = 1$ so $\pi(Z)$ is a line. □

REMARK It is important to mention that Corollary 4.7 and Lemma 4.8 follow the exact same proof strategy as in (Araujo et al., 2023, Lemma 5.90) and (Araujo et al., 2023, Lemma 5.91).

MAIN THEOREM 3. All the smooth threefolds except one in Family №3.12 are K-polystable.

Proof. By Lemma 11.4.7 we know that $\pi(Z)$ is a line. We have that $\pi(Z) \neq L$ (by Lemma 11.4.2), $\pi(Z)$ is not one of the G-invariant lines which does not intersect C (by Lemma 11.4.1) and $\pi(Z)$ is not one of the G-invariant lines which intersect C (by Corollary 11.4.4). So an irreducible curve Z described in Section 11.3.1 does not exist and we came to a contradiction. This argument, alongside (Araujo et al., 2023, §5.18) complete the proof of the Main Theorem. □

Chapter 12

K-stability in Family №3.5

In this chapter we prove K-stability of smooth Fano threefolds in Family №3.5 that satisfy very explicit generality condition. The results presented in this chapter were published in *Annali dell'Universita di Ferrara* (see Denisova (2024a)).

12.1 Calabi Problem for Family №3.5

Let $S = \mathbb{P}^1 \times \mathbb{P}^1$, let C be a smooth curve in S of degree $(5,1)$, and let $\varepsilon\colon C \to \mathbb{P}^1$ be the morphism induced by the projection $S \to \mathbb{P}^1$ to the first factor. Then ε is a finite morphism of degree five, and we may assume that the points $([1:0],[0:1])$ and $([0:1],[1:0])$ are among its ramifications points. This assumption implies that the curve C is given by

$$u\big(x^5 + a_1x^4y + a_2x^3y^2 + a_3x^2y^3\big) = v\big(y^5 + b_1xy^4 + b_2x^2y^3 + b_3x^3y^2\big)$$

for some a_1, a_2, a_3, b_1, b_2, b_3, where $([u:v],[x:y])$ are coordinates on S. Note that the ramification index of the point $([1:0],[0:1])$ can be computed as follows:

$$\begin{cases} 2 \text{ if } a_3 \neq 0, \\ 3 \text{ if } a_3 = 0 \text{ and } a_2 \neq 0, \\ 4 \text{ if } a_3 = a_2 = 0 \text{ and } a_1 \neq 0, \\ 5 \text{ if } a_3 = a_2 = a_1 = 0. \end{cases}$$

Likewise, we can compute the ramification index of the point $([0:1],[1:0])$. We may assume that

- $([1:0],[0:1])$ has the largest ramification index among ramifications points of ε
- the ramification index of the point $([0:1],[1:0])$ is the second largest index.

If both these indices are 5, then $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0$, the morphism ε does not have other ramification points, and the equation of the curve C simplifies as

$$ux^5 = vy^5.$$

In this case, we have $\mathrm{Aut}(S,C) \cong \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$. In all other cases, this group is finite (Cheltsov et al., 2019, Corollary 2.7). Now, we consider embedding $S \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ given by

$$\big([u:v],[x:y]\big) \mapsto \big([u:v],[x^2:xy:y^2]\big),$$

and identify S and C with their images in $\mathbb{P}^1 \times \mathbb{P}^2$. Let $\pi\colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ be the blow up of the curve C. Then X is a smooth Fano threefold in the deformation family № 3.5 in the Mori–Mukai list and every smooth member of this family can be obtained in this way. We know from (Araujo et al., 2023, Section 5.14), that

- X is K-stable if the numbers a_1, a_2, a_3, b_1, b_2, b_3 are general enough,
- X is K-polystable if $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0$.

However, for some a_1, a_2, a_3, b_1, b_2, b_3, the threefold X is not K-polystable.

Example 12.1.1. If $(a_1,a_2,a_3) = (0,0,0) \neq (b_1,b_2,b_3)$, then X is not K-polystable (Araujo et al., 2023, Lemma 7.6).

Note also that it follows from the proof of (Cheltsov et al., 2019, Lemma 8.7) that $\mathrm{Aut}(X) \cong \mathrm{Aut}(S,C)$. In particular, we conclude the group $\mathrm{Aut}(X)$ is finite if and only if $(a_1,a_2,a_3,b_1,b_2,b_3) \neq (0,0,0,0,0,0)$. In this case, the threefold X is K-polystable if and only if it is K-stable. Moreover, we have

Conjecture 12.1.2 (Araujo et al. (2023)). *The Fano threefold X is K-stable if and only if $(a_1,a_2,a_3) \neq (0,0,0)$.*

Geometrically, this conjecture says that the following two conditions are equivalent:

1. the threefold X is K-stable,
2. the morphism $\varepsilon\colon C \to \mathbb{P}^1$ does not have ramification points of ramification index five.

The goal of this paper is to prove the following (slightly weaker) result:

MAIN THEOREM 4. If all ramification points of ε have ramification index two, then X is K-stable.

Let $\mathrm{pr}_1 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ be the projection to the first factor and $\phi_1 = \mathrm{pr}_1 \circ \pi$. Then ϕ_1 is a fibration into del Pezzo surfaces of degree four, and every singular fiber of this fibration has Du Val singular points of types $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$ or $\mathbb{A}_4$, and we have the following possibilities for the singularities of a given singular fiber

1. one singular point of type $\mathbb{A}_1$,
2. two singular points of type $\mathbb{A}_1$,
3. one singular point of type $\mathbb{A}_2$,
4. one singular point of type $\mathbb{A}_1$ and one singular point of type $\mathbb{A}_2$,
5. one singular point of type $\mathbb{A}_3$,
6. one singular point of type $\mathbb{A}_4$.

Note that ϕ_1 has at most two singular fibers that have singular points of type $\mathbb{A}_4$. Moreover, if ϕ_1 has two singular fibers with singular points of type $\mathbb{A}_4$ then all numbers a_i and b_j vanish, so that X is K-polystable. Vice versa, if ϕ_1 has exactly one singular fiber with a point type $\mathbb{A}_1$, then the authors of Araujo et al. (2023) proved that X is not K-polystable. Moreover, they conjectured that X is K-stable in all remaining cases. Now **Main Theorem** and Conjecture 12.1.2 can be restated as follows:

MAIN THEOREM 4. If every singular fiber of ϕ_1 has only singular points of type $\mathbb{A}_1$, then X is K-stable.

Conjecture 12.1.3. *The Fano threefold X is K-stable if and only if every singular fiber of ϕ_1 has only singular points of type $\mathbb{A}_1$, $\mathbb{A}_2$ or $\mathbb{A}_3$.*

12.2 Proof of the Main Theorem (Family №3.5)

Suppose that each singular fiber of the fibration ϕ_1 has one or two singular points of type $\mathbb{A}_1$. Note that this fiber is a del Pezzo surface of degree 4 with Du Val singularities. The Fano threefold X is K-stable if and only if for every prime divisor $\mathbf{F}$ over X we have

$$\beta(\mathbf{F}) = A_X(\mathbf{F}) - S_X(\mathbf{F}) > 0$$

where $A_X(\mathbf{F})$ is the log discrepancy of the divisor $\mathbf{F}$, and

$$S_X\big(\mathbf{F}\big) = \frac{1}{(-K_X)^3}\int\limits_0^\infty \mathrm{vol}\big(-K_X - u\mathbf{F}\big)du.$$

To show this, we fix a prime divisor $\mathbf{F}$ over X. Then we set $Z = C_X(\mathbf{F})$. If Z is an irreducible surface, then it follows from Fujita (2016) that $\beta(\mathbf{F}) > 0$, see also (Araujo et al., 2023, Theorem 3.17). Therefore, we may assume that

- either Z is an irreducible curve in X,
- or Z is a point in X.

In both cases, we fix a point $O \in Z$. Let $\overline{T}$ be the fiber of ϕ_1 which contains O. Then $\overline{T}$ is a del Pezzo surface with at most Du Val singularities. Set

$$\tau(\overline{T}) = \sup\Big\{u \in \mathbb{R}_{>0}\Big|\text{ the divisor } -K_X - u\overline{T} \text{ is pseudo-effective}\Big\}$$

For $u \in [0, \tau(\overline{T})]$ let $P(u)$ be the positive part of the Zariski decomposition of the divisor $-K_X - u\overline{T}$, and let $N(u)$ be its negative part. We denote $\widetilde{S}$ to be the proper transform on X of the surface S. Then we have

$$P(u) = \begin{cases} -K_X - u\overline{T} & \text{if } u \in [0,1], \\ -K_X - u\overline{T} - (u-1)\widetilde{S} & \text{if } u \in [1,2]. \end{cases} \quad \text{and } N(u) = \begin{cases} 0 & \text{if } u \in [0,1], \\ (u-1)\widetilde{S} & \text{if } u \in [1,2]. \end{cases}$$

which gives

$$S_X(\overline{T}) = \frac{1}{20} \int_0^2 P(u)^3 du = \frac{69}{80} < 1$$

Now, for every prime divisor F over the surface $\overline{T}$, we set

$$S\big(W^{\overline{T}}_{\bullet,\bullet}; F\big) = \frac{3}{(-K_X)^3} \int_0^{\tau} \mathrm{ord}_F\big(N(u)|_{\overline{T}}\big)\big(P(u)|_{\overline{T}}\big)^2 du + \frac{3}{(-K_X)^3} \int_0^{\tau}\int_0^{\infty} \mathrm{vol}\big(P(u)\big|_{\overline{T}} - vF\big) dv du.$$

Then, following Abban and Zhuang (2022); Araujo et al. (2023), we let

$$\delta_O\big(\overline{T}, W^{\overline{T}}_{\bullet,\bullet}\big) = \inf_{\substack{F/\overline{T} \\ O \in C_{\overline{T}}(F)}} \frac{A_{\overline{T}}(F)}{S\big(W^{\overline{T}}_{\bullet,\bullet}; F\big)},$$

where the infimum is taken by all prime divisors over the surface $\overline{T}$ whose center on $\overline{T}$ contains O. Then it follows from Abban and Zhuang (2022); Araujo et al. (2023) that

$$\frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\frac{1}{S_X(\overline{T})}, \delta_O\big(\overline{T}, W^{\overline{T}}_{\bullet,\bullet}\big)\right\}.$$

Therefore, if $\beta(\mathbf{F}) \leqslant 0$, then $\delta_O(\overline{T}, W^{\overline{T}}_{\bullet,\bullet}) \leqslant 1$.

Let's prove that $\delta_O(\overline{T}, W^{\overline{T}}_{\bullet,\bullet}) > 1$. To estimate $\delta_O(\overline{T}, W^{\overline{T}}_{\bullet,\bullet})$, we set $\overline{D} = P(u)|_{\overline{T}}$. We have

$$\overline{D} = \begin{cases} -K_{\overline{T}} & \text{if } u \in [0,1], \\ -K_{\overline{T}} - (u-1)\overline{C}_2 & \text{if } u \in [1,2]. \end{cases}$$

where $\overline{C}_2 := \widetilde{S}|_{\overline{T}}$. Then $\overline{D}$ is ample for $u \in [0,2)$, and

$$\overline{D}^2 = \begin{cases} 4 & \text{if } u \in [0,1], \\ 5 - u^2 & \text{if } u \in [1,2]. \end{cases}$$

By (Araujo et al., 2023, Lemma 5.68) and (Araujo et al., 2023, Lemma 5.69) we have

Lemma 12.2.1. *If $O \in \widetilde{S}$ then $\delta_O(X) > 1$.*

Lemma 12.2.2. *If $\overline{T}$ is smooth then $\delta_O(X) > 1$.*

Thus, to prove **Main Theorem**, we may assume that $O \notin \widetilde{S}$ and $\overline{T}$ is singular. Recall that

$$\delta_O\big(\overline{T},\overline{D}\big) = \inf_{\substack{F/\overline{T}\\ O\in C_{\overline{T}}(F)}} \frac{A_{\overline{T}}(F)}{S_{\overline{D}}\big(F\big)}$$

where the infimum is taken by all prime divisors over $\overline{T}$ whose center on $\overline{T}$ contain O, and $S_{\overline{D}}(F) = \frac{1}{\overline{D}^2}\int\limits_0^\infty \mathrm{vol}\big(\overline{D}-vF\big)dv$. Usually $\delta_O(\overline{T},-K_{\overline{T}})$ is denoted by $\delta_O(\overline{T})$.

Note that since $O \notin \widetilde{S}$ then for any divisor F over $\overline{T}$ then we get

$$\begin{aligned}
&S\big(W^{\overline{T}}_{\bullet,\bullet};F\big) = \\
&= \frac{3}{(-K_X)^3}\left(\int_0^\tau \big(P(u)^2\cdot\overline{T}\big)\cdot \mathrm{ord}_O\Big(N(u)\big|_{\overline{T}}\Big)du + \int_0^\tau\int_0^\infty \mathrm{vol}\big(P(u)\big|_{\overline{T}}-vF\big)dvdu\right) = \\
&= \frac{3}{20}\int_0^\tau\int_0^\infty \mathrm{vol}\big(P(u)\big|_{\overline{T}}-vF\big)dvdu = \\
&= \frac{3}{20}\left(\int_0^1\int_0^\infty \mathrm{vol}\big(-K_{\overline{T}}-vF\big)dvdu + \int_1^2\int_0^\infty \mathrm{vol}\big(-K_{\overline{T}}-(u-1)\overline{C}_2-vF\big)dvdu\right) = \\
&= \frac{3}{20}\left(\int_0^\infty \mathrm{vol}\big(-K_{\overline{T}}-vF\big)dv + \int_0^\infty \mathrm{vol}\big(-K_{\overline{T}}-(u-1)\overline{C}_2-vF\big)dv\right) \le \\
&= \frac{3}{20}\left(\int_0^\infty \mathrm{vol}\big(-K_{\overline{T}}-vF\big)dv + \int_0^\infty \mathrm{vol}\big(-K_{\overline{T}}-vF\big)dv\right) = \\
&= \frac{3}{10}\left(\int_0^\infty \mathrm{vol}\big(-K_{\overline{T}}-vF\big)dv\right) = \frac{6}{5}\left(\frac{1}{4}\int_0^\infty \mathrm{vol}\big(-K_{\overline{T}}-vF\big)dv\right) = \frac{6}{5}S_{\overline{T}}(F) \le \frac{6}{5}\cdot\frac{A_{\overline{T}}(F)}{\delta_O(\overline{T})}.
\end{aligned}$$

Thus, if $\delta_O(\overline{T}) > 6/5$, then $\delta_O(\overline{T},W^{\overline{T}}_{\bullet,\bullet}) > 1$. To estimate $\delta_O(\overline{T},W^{\overline{T}}_{\bullet,\bullet})$ in the case when $\delta_O(\overline{T}) \le 6/5$, we define the following positive continuous function on $[1,2]$:

$$f(u) := \begin{cases} \dfrac{15-3u^2}{16+3u-9u^2+2u^3} \text{ if } u\in[1,a], \\ \dfrac{15-3u^2}{11-u^3} \text{ if } u\in[a,2]. \end{cases}$$

where a is a root of $3u^3-9u^2+3u+5$ on $[1,2]$. More precisely, $a\in[1.355,1.356]$. In the appendix we prove that for each O such that $\delta_O(\overline{T}) \le \frac{6}{5}$ we have $\delta_O(\overline{T},\overline{D}) \ge f(u)$ for every $u\in[1,2]$.

So we obtain

$$S\big(W^{\overline{T}}_{\bullet,\bullet};F\big)=\frac{3}{(-K_X)^3}\int\limits_1^2\int\limits_0^{\infty}\mathrm{vol}\big(P(u)\big|_{\overline{T}}-vF\big)dvdu+\frac{3}{(-K_X)^3}\int\limits_0^1\int\limits_0^{\infty}\mathrm{vol}\big(P(u)\big|_{\overline{T}}-vF\big)dvdu\leq$$

$$\leq\frac{3}{20}\left(\int\limits_1^2\frac{(5-u^2)}{\delta_O(\overline{T},\overline{D})}du\right)A_{\overline{T}}(F)+\frac{3}{20}\cdot\frac{4A_{\overline{T}}(F)}{\delta_O(\overline{T})}\leq\frac{3}{20}\left(\int\limits_1^2\frac{(5-u^2)}{f(u)}du\right)A_{\overline{T}}(F)+\frac{3}{5}A_{\overline{T}}(F)\leq$$

$$\leq\frac{3}{20}\left(\int\limits_1^{1.356}(5-u^2)\frac{16+3u-9u^2+2u^3}{15-3u^2}du+\int\limits_{1.355}^{2}(5-u^2)\frac{11-u^3}{15-3u^2}du\right)A_{\overline{T}}(F)+\frac{3}{5}A_{\overline{T}}(F)\leq$$

$$\leq\frac{99}{100}A_{\overline{T}}(F).$$

Thus $\frac{A_{\overline{T}}(F)}{S\left(W^{\overline{T}}_{\bullet,\bullet};F\right)}\geq\frac{100}{99}$ for every prime divisor F over $\overline{T}$ whose support on F contains O, so that $\delta_O(W^{\overline{T}},F)\geq\frac{100}{99}$, which implies $\beta(\mathbf{F})>0$ and X is K-stable.

Remark 12.2.3. If O were a singular point of type $\mathbb{A}_2$ in $\overline{T}$, this approach would not work, because as is shown in Section 12.3.3 we have $\delta_O(\overline{T},\overline{D})=\frac{15-3u^2}{u^3-6u^2+19}$ and there is prime divisor F over $\overline{T}$ such that $A_{\overline{T}}(F)=1$ and $S(W^{\overline{T}}_{\bullet,\bullet};F)=\frac{83}{80}$, which implies that $\delta_O(\overline{T},W^{\overline{T}}_{\bullet,\bullet})\leq\frac{80}{83}$.

12.3 Polarized δ-invariants via Kento Fujita's formulas

Let us use the notations from the previous section. Recall that $\overline{T}$ is a Du Val del Pezzo surface, and the blow-up π induces a birational morphism $\upsilon:\overline{T}\to\mathbb{P}^2$. We assume that $\overline{T}$ is singular. We have the following commutative diagram:

$$\begin{array}{ccc} & T & \\ {}^{\sigma}\swarrow & & \searrow^{\eta} \\ \overline{T} & \xrightarrow{\ \upsilon\ } & \mathbb{P}^2\end{array}$$

Suppose that $u\in[1,2]$. Recall that $\overline{D}=-K_{\overline{T}}-(1-u)\overline{C}_2$. Observe that $\overline{C}_2$ is contained in the smooth locus of the surface $\overline{T}$. Let C_2 be the strict transform of the curve $\overline{C}_2$ on the surface T, and set $D=-K_T-(1-u)C_2$. Note that $D=\sigma^*(\overline{D})$, so the divisor D is big and nef for $u\in[1,2]$. Recall

$$\delta_O(\overline{T},\overline{D})=\inf_{\substack{F/\overline{T}\\ O\in C_{\overline{T}}(F)}}\frac{A_{\overline{T}}(F)}{S_D(F)},$$

where the infimum is taken over all prime divisors F over $\overline{T}$ such that $O\in C_{\overline{T}}(F)$. For every point $P\in T$, we also define

$$\delta_P(T,D)=\inf_{\substack{E/T\\ P\in C_T(E)}}\frac{A_T(E)}{S_D(E)},$$

where the infimum is taken over all prime divisors E over T such that $P \in C_T(E)$. Since $D = \sigma^*(\overline{D})$ and $K_T = \sigma^*(K_{\overline{T}})$, we have

$$\delta_O(\overline{T}, \overline{D}) = \inf_{\substack{P \in T \\ \sigma(P) = O}} \delta_P(T, D).$$

Thus, to estimate $\delta_O(\overline{T}, \overline{D})$, it is enough to estimate $\delta_P(T, D)$ for all points P such that $\sigma(P) = O$. Let $\mathscr{C}$ be a smooth curve on T containing P. Set

$$\tau(\mathscr{C}) = \sup\left\{ v \in \mathbb{R}_{\geqslant 0} \mid \text{the divisor } D - v\mathscr{C} \text{ is pseudo-effective} \right\}.$$

For $v \in [0, \tau(\mathscr{C})]$, let $P(v)$ and $N(v)$ denote the positive and negative parts, respectively, of the Zariski decomposition of $D - v\mathscr{C}$. Then we set

$$S\big(W^{\mathscr{C}}_{\bullet,\bullet}; P\big) = \frac{2}{D^2} \int_0^{\tau(\mathscr{C})} h_D(v)\, dv,$$

where

$$h_D(v) = (P(v) \cdot \mathscr{C}) \cdot (N(v) \cdot \mathscr{C})_P + \frac{(P(v) \cdot \mathscr{C})^2}{2}.$$

It follows from Abban and Zhuang (2022); Araujo et al. (2023) that

$$\delta_P(T, D) \geqslant \min\left\{ \frac{1}{S_D(\mathscr{C})}, \frac{1}{S(W^{\mathscr{C}}_{\bullet,\bullet}; P)} \right\}.$$

We will estimate $\delta_P(T, D)$ in the following, using the notations above, for a suitable choice of the curve $\mathscr{C}$, the threshold $\tau(\mathscr{C})$, and the decompositions $P(v)$ and $N(v)$ in specific cases.

A similar approach was taken in Belousov and Loginov (2024) and Cheltsov (2024).

12.3.1 Du Val Del Pezzo surface of degree 4 with $\mathbb{A}_1$ singularity

Suppose that $\overline{T}$ has one singular point and this point is a singular point of type $\mathbb{A}_1$. Then η is a blow up of $\mathbb{P}^2$ at points P_1, P_2, P_3 and P_4 in general position and a point P_5 which belongs to the exceptional divisor corresponding to P_4 and no other negative curve. Suppose $\mathbf{E} := L_{14} \cup L_{24} \cup L_{24} \cup E_5$. In Part I we proved:

$$\delta_P(T) = \begin{cases} 1 \text{ if } P \in E_4, \\ 6/5 \text{ if } P \in \mathbf{E} \backslash E_4, \\ 4/3 \text{ if } P \text{ belongs to two curves in } \{E_1, E_2, E_3, L_{12}, L_{13}, L_{23}, L_{45}, C_2\}, \\ 18/13 \text{ if } P \text{ belongs to exactly one curve in } \{E_1, E_2, E_3, L_{12}, L_{13}, L_{23}, L_{45}, C_2\} \backslash \mathbf{E}, \\ 3/2 \text{ otherwise.} \end{cases}$$

where E_1, E_2, E_3, E_4, E_5 are exceptional divisors corresponding to P_1, P_2, P_3, P_4, P_5 respectively, C_2 is a strict transform of a (-1)-curve coming from the conic on $\mathbb{P}^2$, L_{ij} are strict transforms of the lines passing through P_i and P_j for $(i,j) \in \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$ and L_{45} a strict transform of a (-1)-curve coming from a line on $\mathbb{P}^2$. The dual graph of (-1) and (-2)-curves is given in the following picture:

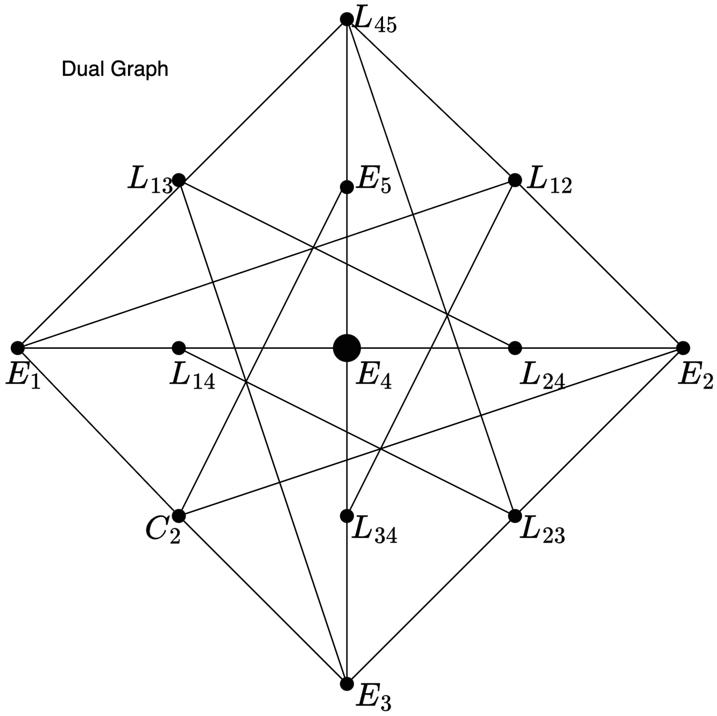


Figure 12.1: Polarized δ-invariants: $\mathbb{A}_1$ singularity

Lemma 12.3.1. *Suppose P is a point on T and $D = -K_T - (u-1)C_2$ with $D^2 = 5 - u^2$ then*

$$\delta_P(T,D) = \begin{cases} \dfrac{15-3u^2}{16+3u-9u^2+2u^3} & \text{for } P \in E_4 \backslash E_5 \text{ and } u \in [1,2], \\ \dfrac{15-3u^2}{11-u^3} & \text{for } P \in E_5 \backslash E_4 \text{ and } u \in [1,2], \\ \dfrac{15-3u^2}{3u^3-12u^2+6u+13} & \text{for } P \in L_{14} \backslash (E_4 \cup E_5 \cup E_1) \text{ and } u \in [1,2]. \end{cases}$$

and

$$\delta_P(T,D) \geq \begin{cases} \dfrac{15-3u^2}{3u^3-12u^2+6u+13} & \text{for } P = E_4 \cap E_5 \text{ and } u \in [1,a], \\ \dfrac{15-3u^2}{11-u^3} & \text{for } P = E_4 \cap E_5 \text{ and } u \in [a,2]. \end{cases}$$

and

$$\delta_P(T,D) \ge \begin{cases} \frac{15-3u^2}{3u^3-12u^2+6u+13} & \text{for } P = L_{14} \cap E_1 \text{ and } u \in [1,b], \\ \frac{2(15-3u^2)}{19-2u^3} & \text{for } P = L_{14} \cap E_1 \text{ and } u \in [b,3/2], \\ \frac{15-3u^2}{3u^3-18u^2+27u-4} & \text{for } P = L_{14} \cap E_1 \text{ and } u \in [3/2,2]. \end{cases}$$

where a is a root of $3u^3-9u^2+3u+5$ on $[1,2]$, b is a root of $8u^3-24u^2+12u+7$ on $[1,3/2]$. Note that $a \in [1.355,1.356]$, $b \in [1.261,1.262]$.

Proof. **Step 1.** Suppose $P \in E_4$. In this case we set $\mathscr{C} = E_4$. Then $\tau(\mathscr{C}) = 3-u$. The Zariski Decomposition of the divisor $D - vE_4$ is given by:

$$P(v) = \begin{cases} -K_T - (u-1)C_2 - vE_4 \text{ for } v \in [0,2-u], \\ -K_T - (u-1)C_2 - vE_4 - (u+v-2)E_5 \text{ for } v \in [2-u,1], \\ -K_T - (u-1)C_2 - vE_4 - (u+v-2)E_5 - (v-1)(L_{14}+L_{24}+L_{34}) \text{ for } v \in [1,3-u]. \end{cases}$$

and

$$N(v) = \begin{cases} 0 \text{ for } v \in [0,2-u], \\ (u+v-2)E_5 \text{ for } v \in [2-u,1], \\ (u+v-2)E_5 + (v-1)(L_{14}+L_{24}+L_{34}) \text{ for } v \in [1,3-u]. \end{cases}$$

Moreover,

$$P(v)^2 = \begin{cases} 5-u^2-2v^2 \text{ for } v \in [0,2-u], \\ 9+2uv-4u-4v-v^2 \text{ for } v \in [2-u,1], \\ 2(2-v)(3-u-v) \text{ for } v \in [1,3-u]. \end{cases}$$

and

$$P(v) \cdot \mathscr{C} = \begin{cases} 2v \text{ for } v \in [0,2-u], \\ 2-u+v \text{ for } v \in [2-u,1], \\ 5-u-2v \text{ for } v \in [1,3-u]. \end{cases}$$

Thus,

$$S_D(\mathscr{C}) = \frac{1}{5-u^2}\Big(\int_0^{2-u} 5-u^2-2v^2dv + \int_{2-u}^1 9+2uv-4u-4v-v^2dv + \int_1^{3-u} 2(2-v)(3-u-v)dv\Big) = \frac{16+3u-9u^2+2u^3}{15-3u^2}.$$

Thus, $\delta_P(T,D) \le \frac{15-3u^2}{16+3u-9u^2+2u^3}$ for $P \in E_4$. Note that we have:

- if $P\in E_4\backslash(E_5\cup L_{14}\cup L_{24}\cup L_{34})$

$$h_D(v)=\begin{cases}2v^2 \text{ for } v\in[0,2-u],\\ \frac{(2-u+v)^2}{2} \text{ for } v\in[2-u,1],\\ \frac{(5-u-2v)^2}{2} \text{ for } v\in[1,3-u].\end{cases}$$

- if $P=E_4\cap E_5$

$$h_D(v)=\begin{cases}2v^2 \text{ for } v\in[0,2-u],\\ \frac{(2-u+v)(u+3v-2)}{2} \text{ for } v\in[2-u,1],\\ \frac{(u+1)(5-u-2v)}{2} \text{ for } v\in[1,3-u]\end{cases}$$

- if $P\in E_4\cap(L_{14}\cup L_{24}\cup L_{34})$

$$h_D(v)=\begin{cases}2v^2 \text{ for } v\in[0,2-u],\\ \frac{(2-u+v)^2}{2} \text{ for } v\in[2-u,1],\\ \frac{(3-u)(5-u-2v)}{2} \text{ for } v\in[1,3-u].\end{cases}$$

So we have

- if $P\in E_4\backslash(E_5\cup L_{14}\cup L_{24}\cup L_{34})$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P)=\frac{2}{5-u^2}\Big(\int_0^{2-u}2v^2dv+\int_{2-u}^1\frac{(2-u+v)^2}{2}dv+\int_1^{3-u}\frac{(5-u-2v)^2}{2}dv\Big)=$$
$$=\frac{9+6u-9u^2+2u^3}{15-3u^2}\le\frac{16+3u-9u^2+2u^3}{15-3u^2}.$$

- if $P=E_4\cap E_5$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P)=\frac{2}{5-u^2}\Big(\int_0^{2-u}2v^2dv+\int_{2-u}^1\frac{(2-u+v)(u+3v-2)}{2}dv+$$
$$+\int_1^{3-u}\frac{(u+1)(5-u-2v)}{2}dv\Big)=\frac{11-u^3}{15-3u^2}.$$

- if $P\in E_4\cap(L_{14}\cup L_{24}\cup L_{34})$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P)=\frac{2}{5-u^2}\Big(\int_0^{2-u}2v^2dv+\int_{2-u}^1\frac{(2-u+v)^2}{2}dv+$$
$$+\int_1^{3-u}\frac{(3-u)(5-u-2v)}{2}dv\Big)=$$
$$=\frac{13+3u^3-12u^2+6u}{15-3u^2}\le\frac{16+3u-9u^2+2u^3}{15-3u^2}.$$

We obtain that

$$\delta_P(T,D) = \frac{15-3u^2}{16+3u-9u^2+2u^3} \text{ for } P \in E_4 \backslash E_5 \text{ and } u \in [1,2].$$

and

$$\delta_P(T,D) \geq \begin{cases} \dfrac{15-3u^2}{16+3u-9u^2+2u^3} \text{ for } P = E_4 \cap E_5 \text{ and } u \in [1,a], \\ \dfrac{15-3u^2}{11-u^3} \text{ for } P = E_4 \cap E_5 \text{ and } u \in [a,2]. \end{cases}$$

where a is a root of $3u^3-9u^2+3u+5$ on $[1,2]$. Note that $a \in [1.355, 1.356]$.

Step 2. Suppose $P \in E_5$. In this case we set $\mathscr{C} = E_5$. Then $\tau(\mathscr{C}) = 2$. The Zariski Decomposition of the divisor $D - vE_5$ is given by:

$$P(v) = \begin{cases} -K_T - (u-1)C_2 - vE_5 - \frac{v}{2}E_4 \text{ for } v \in [0,1], \\ -K_T - (u-1)C_2 - vE_5 - \frac{v}{2}E_4 - (v-1)L_{45} \text{ for } v \in [1,u], \\ -K_T - (u-1)C_2 - vE_5 - \frac{v}{2}E_4 - (v-1)L_{45} - (v-u)C_2 \text{ for } v \in [u,2]. \end{cases}$$

and

$$N(v) = \begin{cases} \frac{v}{2}E_4 \text{ for } v \in [0,1], \\ \frac{v}{2}E_4 + (v-1)L_{45} \text{ for } v \in [1,u], \\ \frac{v}{2}E_4 + (v-1)L_{45} + (v-u)C_2 \text{ for } v \in [u,2]. \end{cases}$$

Moreover,

$$P(v)^2 = \begin{cases} 5 - 4v + 2uv - u^2 - \frac{v^2}{2} \text{ for } v \in [0,1], \\ 6 - 6v + \frac{v^2}{2} + 2uv - u^2 \text{ for } v \in [1,u], \\ \frac{3(2-v)^2}{2} \text{ for } v \in [u,2]. \end{cases}$$

and

$$P(v) \cdot \mathscr{C} = \begin{cases} 2 - u + v/2 \text{ for } v \in [0,1], \\ 3 - u - v/2 \text{ for } v \in [1,u], \\ 3 - 3v/2 \text{ for } v \in [u,2]. \end{cases}$$

Thus,

$$S_D(\mathscr{C}) = \frac{1}{5-u^2}\Big(\int_0^1 5 - 4v + 2uv - u^2 - \frac{v^2}{2}dv + \int_1^u 6 - 6v + \frac{v^2}{2} + 2uv - u^2 dv + \\ + \int_u^2 \frac{3(2-v)^2}{3}dv\Big) = \frac{11-u^3}{15-3u^2}.$$

Thus, $\delta_P(T,D) \leq \frac{15-3u^2}{11-u^3}$ for $P \in E_5$. Note that we have:

- if $P \in E_5 \backslash (E_4 \cup C_2 \cup L_{45})$ then

$$h_D(v) = \begin{cases} \frac{(2-u+v/2)^2}{2} \text{ for } v \in [0,1], \\ \frac{(3-u-v/2)^2}{2} \text{ for } v \in [1,u], \\ \frac{(3-3v/2)^2}{2} \text{ for } v \in [u,2]. \end{cases}$$

- if $P = E_5 \cap C_2$ then

$$h_D(v) = \begin{cases} \frac{(2-u+v/2)^2}{2} \text{ for } v \in [0,1], \\ \frac{(3-u-v/2)^2}{2} \text{ for } v \in [1,u], \\ \frac{3(2-v)(6-4u+v)}{8} \text{ for } v \in [u,2]. \end{cases}$$

- if $P = E_5 \cap L_{45}$ then

$$h_D(v) = \begin{cases} \frac{(2-u+v/2)^2}{2} \text{ for } v \in [0,1], \\ \frac{(6-2u-v)(2-2u+3v)}{8} \text{ for } v \in [1,u], \\ \frac{3(2-v)(v+2)}{8} \text{ for } v \in [u,2]. \end{cases}$$

So we have

- if $P \in E_5 \backslash (E_4 \cup C_2 \cup L_{45})$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P) = \frac{2}{5-u^2}\Big(\int_0^1 \frac{(2-u+v/2)^2}{2}dv + \int_1^u \frac{(3-u-v/2)^2}{2}dv + \\ + \int_u^2 \frac{(3-3v/2)^2}{2}dv\Big) = \frac{21+6u-18u^2+5u^3}{2(15-3u^2)} \leq \frac{11-u^3}{15-3u^2}.$$

- if $P = E_5 \cap C_2$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P) = \frac{2}{5-u^2}\Big(\int_0^1 \frac{(2-u+v/2)^2}{2}dv + \int_1^u \frac{(3-u-v/2)^2}{2}dv + \\ + \int_u^2 \frac{3(2-v)(6-4u+v)}{8}dv\Big) = \frac{45-30u+2u^3}{2(15-3u^2)} \leq \frac{11-u^3}{15-3u^2}.$$

- if $P = E_5 \cap L_{45}$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P) = \frac{2}{5-u^2}\Big(\int_0^1 \frac{(2-u+v/2)^2}{2}dv + \int_1^u \frac{(6-2u-v)(2-2u+3v)}{8}dv + \\ + \frac{3(2-v)(v+2)}{8}dv\Big) = \frac{26-12u^2+3u^3}{2(15-3u^2)} \leq \frac{11-u^3}{15-3u^2}.$$

We obtain that

$$\delta_P(T,D) = \frac{15-3u^2}{11-u^3} \text{ for } P \in E_5 \backslash E_4 \text{ and } u \in [1,2].$$

Step 3.1. Suppose $P \in L_{14} \cup L_{24} \cup L_{34}$ and $u \in [1,3/2]$. Without loss of generality, we can assume that $P \in L_{14}$. In this case we set $\mathscr{C} = L_{14}$. Then $\tau(\mathscr{C}) = 3-u$. The Zariski Decomposition of the divisor $D - vL_{14}$ is given by:

$$P(v) = \begin{cases} D - vL_{14} - \frac{v}{2}E_4 \text{ for } v \in [0, 2-u], \\ D - vL_{14} - \frac{v}{2}E_4 - (u+v-2)E_1 \text{ for } v \in [2-u, 1], \\ D - vL_{14} - \frac{v}{2}E_4 - (u+v-2)E_1 - (v-1)L_{23} \text{ for } v \in [1, 4-2u], \\ D - vL_{14} - (u+v-2)(E_1+E_4) - (v-1)L_{23} - (2u+v-4)E_5 \text{ for } v \in [4-2u, 3-u]. \end{cases}$$

and

$$N(v) = \begin{cases} \frac{v}{2}E_4 \text{ for } v \in [0, 2-u], \\ \frac{v}{2}E_4 + (u+v-2)E_1 \text{ for } v \in [2-u, 1], \\ \frac{v}{2}E_4 + (u+v-2)E_1 + (v-1)L_{23} \text{ for } v \in [1, 4-2u], \\ (u+v-2)(E_1+E_4) + (v-1)L_{23} + (2u+v-4)E_5 \text{ for } v \in [4-2u, 3-u]. \end{cases}$$

Moreover

$$P(v)^2 = \begin{cases} 5 - 2v - \frac{v^2}{2} - u^2 \text{ for } v \in [0, 2-u], \\ 9 - 4u - 6v + \frac{v^2}{2} + 2uv \text{ for } v \in [2-u, 1], \\ \frac{(v-2)(3v+4u-10)}{2} \text{ for } v \in [1, 4-2u], \\ 2(u+v-3)^2 \text{ for } v \in [4-2u, 3-u]. \end{cases}$$

and

$$P(v) \cdot \mathscr{C} = \begin{cases} v/2 + 1 \text{ for } v \in [0, 2-u], \\ 3 - u - v/2 \text{ for } v \in [2-u, 1], \\ 4 - u - 3v/2 \text{ for } v \in [1, 4-2u], \\ 2(3-u-v) \text{ for } v \in [4-2u, 3-u]. \end{cases}$$

Thus,

$$S_D(\mathscr{C}) = \frac{1}{5-u^2}\Big(\int_0^{2-u} 5 - 2v - \frac{v^2}{2} - u^2 dv + \int_{2-u}^{1} 9 - 4u - 6v + \frac{v^2}{2} + 2uvdv + \\ + \int_1^{4-2u} \frac{(v-2)(3v+4u-10)}{2} dv + \int_{4-2u}^{3-u} 2(u+v-3)^2 dv\Big) = \frac{3u^3 - 12u^2 + 6u + 13}{15 - 3u^2}.$$

Thus, $\delta_P(T,D) \leq \frac{15-3u^2}{3u^3-12u^2+6u+13}$ for $P \in L_{14}$. Note that we have:

- if $P\in L_{14}\backslash(E_4\cup E_1\cup L_{23}\cup E_5)$ then

$$h_D(v)=\begin{cases}\frac{(v/2+1)^2}{2} \text{ for } v\in[0,2-u],\\ \frac{(3-u-v/2)^2}{2} \text{ for } v\in[2-u,1],\\ \frac{(4-u-3v/2)^2}{2} \text{ for } v\in[1,4-2u],\\ 2(3-u-v)^2 \text{ for } v\in[4-2u,3-u].\end{cases}$$

- if $P=L_{14}\cap E_1$ then

$$h_D(v)=\begin{cases}\frac{(v/2+1)^2}{2} \text{ for } v\in[0,2-u],\\ \frac{(6-2u-v)(2u+3v-2)}{8} \text{ for } v\in[2-u,1],\\ \frac{(8-2u-3v)(2u+v)}{8} \text{ for } v\in[1,4-2u],\\ (3-u-v) \text{ for } v\in[4-2u,3-u].\end{cases}$$

- if $P=L_{14}\cap L_{23}$ then

$$h_D(v)=\begin{cases}\frac{(v/2+1)^2}{2} \text{ for } v\in[0,2-u],\\ \frac{(3-u-v/2)^2}{2} \text{ for } v\in[2-u,1],\\ \frac{(8-2u-3v)(4-2u+v)}{8} \text{ for } v\in[1,4-2u],\\ 2(2-u)(3-u-v) \text{ for } v\in[4-2u,3-u].\end{cases}$$

So we have

- if $P\in L_{14}\backslash(E_4\cup E_1\cup L_{23}\cup E_5)$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P)=\frac{2}{5-u^2}\Big(\int_0^{2-u}\frac{(v/2+1)^2}{2}dv+\int_{2-u}^1\frac{(3-u-v/2)^2}{2}dv+ +\int_1^{4-2u}\frac{(4-u-3v/2)^2}{2}dv+\int_{4-2u}^{3-u}2(3-u-v)^2dv\Big)= =\frac{21-u^3-6u}{2(15-3u^2)}\leq\frac{3u^3-12u^2+6u+13}{15-3u^2}.$$

- if $P=L_{14}\cap E_1$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P)=\frac{2}{5-u^2}\Big(\int_0^{2-u}\frac{(v/2+1)^2}{2}dv+\int_{2-u}^1\frac{(6-2u-v)(2u+3v-2)}{8}dv+ +\int_1^{4-2u}\frac{(8-2u-3v)(2u+v)}{8}dv+\int_{4-2u}^{3-u}(3-u-v)dv\Big)=\frac{19-2u^3}{2(15-3u^2)}.$$

- if $P=L_{14}\cap L_{23}$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P)=\frac{2}{5-u^2}\Big(\int_0^{2-u}\frac{(v/2+1)^2}{2}dv+\int_{2-u}^{1}\frac{(3-u-v/2)^2}{2}dv+$$
$$+\int_1^{4-2u}\frac{(8-2u-3v)(4-2u+v)}{8}dv+\int_{4-2u}^{3-u}2(2-u)(3-u-v)dv\Big)=$$
$$=\frac{26-12u^2+3u^3}{2(15-3u^2)}\leq\frac{3u^3-12u^2+6u+13}{15-3u^2}.$$

We obtain that

$$\delta_P(T,D)=\frac{15-3u^2}{3u^3-12u^2+6u+13}\text{ for }P\in L_{14}\backslash(E_4\cup E_5\cup E_1)\text{ and }u\in[1,3/2].$$

and

$$\delta_P(T,D)\geq\begin{cases}\dfrac{15-3u^2}{3u^3-12u^2+6u+13}\text{ for }P=L_{14}\cap E_1\text{ and }u\in[1,b],\\ \dfrac{2(15-3u^2)}{19-2u^3}\text{ for }P=L_{14}\cap E_1\text{ and }u\in[b,3/2].\end{cases}$$

where b is a root of $8u^3-24u^2+12u+7$ on $[1,3/2]$. Note that $b\in[1.261,1.262]$.

Step 3.2. Suppose $P\in L_{14}\cup L_{24}\cup L_{34}$ and $u\in[3/2,2]$. Without loss of generality, we can assume that $P\in L_{14}$. In this case we set $\mathscr{C}=L_{14}$. Then $\tau(\mathscr{C})=3-u$. The Zariski Decomposition of the divisor $D-vL_{14}$ is given by:

$$P(v)=\begin{cases}D-vL_{14}-\frac{v}{2}E_4\text{ for }v\in[0,2-u],\\ D-vL_{14}-\frac{v}{2}E_4-(u+v-2)E_1\text{ for }v\in[2-u,4-2u],\\ D-vL_{14}-(u+v-2)(E_1+E_4)-(2u+v-4)E_5\text{ for }v\in[4-2u,1],\\ D-vL_{14}-(u+v-2)(E_1+E_4)-(v-1)L_{23}-(2u+v-4)E_5\text{ for }v\in[1,3-u].\end{cases}$$

and

$$N(v)=\begin{cases}\frac{v}{2}E_4\text{ for }v\in[0,2-u],\\ \frac{v}{2}E_4+(u+v-2)E_1\text{ for }v\in[2-u,4-2u],\\ (u+v-2)(E_1+E_4)+(2u+v-4)E_5\text{ for }v\in[4-2u,1],\\ (u+v-2)(E_1+E_4)+(v-1)L_{23}+(2u+v-4)E_5\text{ for }v\in[1,3-u].\end{cases}$$

Moreover

$$P(v)^2=\begin{cases}5-2v-\frac{v^2}{2}-u^2\text{ for }v\in[0,2-u],\\ 9-4u-6v+\frac{v^2}{2}+2uv\text{ for }v\in[2-u,4-2u],\\ 2u^2+4uv+v^2-12u-10v+17\text{ for }v\in[4-2u,1],\\ 2(u+v-3)^2\text{ for }v\in[1,3-u].\end{cases}$$

and

$$P(v)\cdot\mathscr{C}=\begin{cases}1+v/2 \text{ for } v\in[0,2-u],\\ 3-u-v/2 \text{ for } v\in[2-u,4-2u],\\ 5-2u-v \text{ for } v\in[4-2u,1],\\ 2(3-u-v) \text{ for } v\in[1,3-u].\end{cases}$$

Thus,

$$S_D(\mathscr{C})=\frac{1}{5-u^2}\Big(\int_0^{2-u}5-2v-\frac{v^2}{2}-u^2dv+\int_{2-u}^{4-2u}9-4u-6v+\frac{v^2}{2}+2uvdv+$$
$$+\int_{4-2u}^{1}2u^2+4uv+v^2-12u-10v+17dv+\int_1^{3-u}2(u+v-3)^2dv\Big)=$$
$$=\frac{3u^3-12u^2+6u+13}{15-3u^2}.$$

Thus, $\delta_P(T,D)\leq\frac{15-3u^2}{3u^3-12u^2+6u+13}$ for $P\in L_{14}$. Note that we have:

- if $P\in L_{14}\backslash(E_4\cup E_1\cup L_{23}\cup E_5)$ then

$$h_D(v)=\begin{cases}\frac{(1+v/2)^2}{2} \text{ for } v\in[0,2-u],\\ \frac{(3-u-v/2)^2}{2} \text{ for } v\in[2-u,4-2u],\\ \frac{(5-2u-v)^2}{2} \text{ for } v\in[4-2u,1],\\ 2(3-u-v)^2 \text{ for } v\in[1,3-u].\end{cases}$$

- if $P=L_{14}\cap E_1$ then

$$h_D(v)=\begin{cases}\frac{(1+v/2)^2}{2} \text{ for } v\in[0,2-u],\\ \frac{(6-2u-v)(2u+3v-2)}{8} \text{ for } v\in[2-u,4-2u],\\ \frac{(v+1)(5-2u-v)}{2} \text{ for } v\in[4-2u,1],\\ 2(3-u-v) \text{ for } v\in[1,3-u].\end{cases}$$

- if $P=L_{14}\cap L_{23}$ then

$$h_D(v)=\begin{cases}\frac{(1+v/2)^2}{2} \text{ for } v\in[0,2-u],\\ \frac{(3-u-v/2)^2}{2} \text{ for } v\in[2-u,4-2u],\\ \frac{(5-2u-v)^2}{2} \text{ for } v\in[4-2u,1],\\ 2(2-u)(3-u-v) \text{ for } v\in[1,3-u].\end{cases}$$

- if $P\in L_{14}\backslash(E_4\cup E_1\cup L_{23}\cup E_5)$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P)=\frac{2}{5-u^2}\Big(\int_0^{2-u}\frac{(v/2+1)^2}{2}dv+\int_{2-u}^{4-2u}\frac{(3-u-v/2)^2}{2}dv+$$
$$+\int_{4-2u}^{1}\frac{(5-2u-v)^2}{2}dv+\int_1^{3-u}2(3-u-v)^2dv\Big)=$$
$$=\frac{7u^3-36u^2+48u-6}{2(15-3u^2)}\leq\frac{3u^3-12u^2+6u+13}{15-3u^2}.$$

- if $P=L_{14}\cap E_1$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P)=\frac{2}{5-u^2}\Big(\int_0^{2-u}\frac{(v/2+1)^2}{2}dv+\int_{2-u}^{4-2u}\frac{(6-2u-v)(2u+3v-2)}{8}dv+$$
$$+\int_{4-2u}^{1}\frac{(v+1)(5-2u-v)}{2}dv+\int_1^{3-u}2(3-u-v)dv\Big)=$$
$$=\frac{3u^3-18u^2+27u-4}{15-3u^2}.$$

- if $P=L_{14}\cap L_{23}$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P)=\frac{2}{5-u^2}\Big(\int_0^{2-u}\frac{(v/2+1)^2}{2}dv+\int_{2-u}^{4-2u}\frac{(3-u-v/2)^2}{2}dv+$$
$$+\int_{4-2u}^{1}\frac{(5-2u-v)^2}{2}dv+\int_1^{3-u}2(2-u)(3-u-v)dv\Big)=$$
$$=\frac{3u^3-12u^2+26}{2(15-3u^2)}\leq\frac{3u^3-12u^2+6u+13}{15-3u^2}.$$

We obtain that

$$\delta_P(T,D)=\frac{15-3u^2}{3u^3-12u^2+6u+13}\text{ for }P\in L_{14}\backslash(E_1\cup E_4\cup E_5)\text{ and }u\in[3/2,2].$$

and

$$\delta_P(T,D)\geq\frac{15-3u^2}{3u^3-18u^2+27u-4}\text{ for }P=L_{14}\cap E_1\text{ and }u\in[3/2,2].$$

□

Corollary 12.3.2. *Let P be a point in T that is contained in $L_{12}\cup L_{24}\cup L_{34}\cup E_4\cup E_5$ then*

$$\delta_P(T,D)\geq\begin{cases}\dfrac{15-3u^2}{16+3u-9u^2+2u^3}\text{ for }u\in[1,a],\\ \dfrac{15-3u^2}{11-u^3}\text{ for }u\in[a,2].\end{cases}$$

Corollary 12.3.3. *Suppose O is a point on a del Pezzo surface $\overline{T}$ with $\mathbb{A}_1$ singularity and $\delta_O(T)\leq\frac{6}{5}$ then*

$$\delta_O(\overline{T},\overline{D})\geq\begin{cases}\dfrac{15-3u^2}{16+3u-9u^2+2u^3}\text{ for }u\in[1,a],\\ \dfrac{15-3u^2}{11-u^3}\text{ for }u\in[a,2].\end{cases}$$

12.3.2 Du Val Del Pezzo surface of degree 4 with $2\mathbb{A}_1$ singularities

Suppose that $\overline{T}$ has two singular points and these points are singular point of type $\mathbb{A}_1$. Then η is a blow up of $\mathbb{P}^2$ at points P_1, P_2, and P_4 in general position and after that blowing up a point P_3 which belongs to the exceptional divisor corresponding to P_2 and a point P_5 which belongs to the exceptional divisor corresponding to P_4 and no other negative curve. In Part I we proved that:

$$\delta_P(T) = \begin{cases} 1 \text{ if } P \in (E_2 \cup E_4 \cup L_{24}), \\ 6/5 \text{ if } P \in (E_3 \cup E_5 \cup L_{12} \cup L_{14}) \backslash (E_2 \cup E_4), \\ 4/3 \text{ if } P \in (C_2 \cap E_1) \cup (L_{23} \cap L_{45}), \\ 18/13 \text{ if } P \in (C_2 \cup E_1 \cup L_{23} \cup L_{45}) \backslash \big((C_2 \cap E_1) \cup (L_{23} \cap L_{45}) \cup (E_3 \cup E_5 \cup L_{12} \cup L_{14}) \big), \\ 3/2 \text{ otherwise.} \end{cases}$$

where E_1, E_2, E_3, E_4, E_5 are exceptional divisors corresponding to P_1, P_2, P_3, P_4, P_5 respectively, C_2 is a strict transform of a (-1)-curve coming from the conic on $\mathbb{P}^2$, L_{ij} are strict transforms of the lines passing through P_i and P_j for $(i,j) \in \{(1,2),(1,4)\}$ and L_{45} and L_{23} are strict transforms of a (-1)-curve coming from lines passing through P_2 and P_4 respectively on $\mathbb{P}^2$. The dual graph of (-1) and (-2)-curves is given in the following picture:

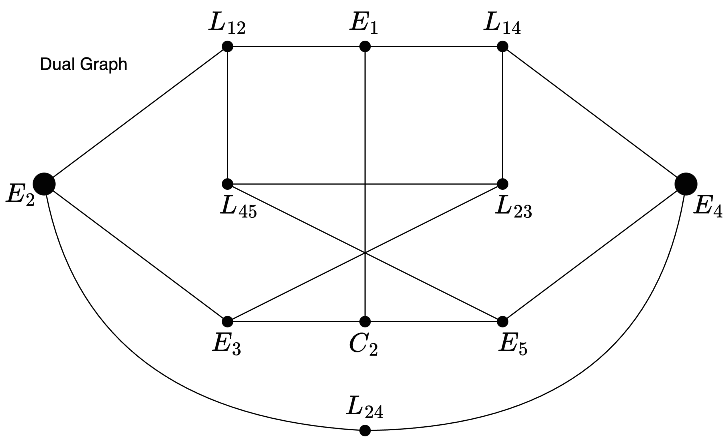


Figure 12.2: Polarized δ-invariants: $2\mathbb{A}_1$ singularities

Lemma 12.3.4. *Suppose P is a point on T and $D = -K_T - (u-1)C_2$ with $D^2 = 5 - u^2$ then*

$$\delta_P(T,D) = \begin{cases} \dfrac{15-3u^2}{16+3u-9u^2+2u^3} \text{ for } P \in (E_2 \cup E_4) \backslash (E_3 \cup E_5) \text{ and } u \in [1,2], \\ \dfrac{15-3u^2}{11-u^3} \text{ for } P \in (E_3 \cup E_5) \backslash (E_2 \cup E_4) \text{ and } u \in [1,2], \\ \dfrac{15-3u^2}{u^3-6u^2+6u+5} \text{ for } P \in L_{24} \backslash (E_2 \cup E_4) \text{ and } u \in [1,2], \\ \dfrac{15-3u^2}{3u^3-12u^2+6u+13} \text{ for } P \in L_{14} \backslash (E_4 \cup E_5 \cup E_1) \text{ and } u \in [1,2]. \end{cases}$$

and

$$\delta_P(T,D)\geq\begin{cases}\dfrac{15-3u^2}{16+3u-9u^2+2u^3} \text{ for } P\in\{E_2\cap E_3,E_4\cap E_5\} \text{ and } u\in[1,a],\\ \dfrac{15-3u^2}{11-u^3} \text{ for } P\in\{E_2\cap E_3,E_4\cap E_5\} \text{ and } u\in[a,2].\end{cases}$$

and

$$\delta_P(T,D)\geq\begin{cases}\dfrac{15-3u^2}{3u^3-12u^2+6u+13} \text{ for } P=L_{14}\cap E_1 \text{ and } u\in[1,b],\\ \dfrac{2(15-3u^2)}{19-2u^3} \text{ for } P=L_{14}\cap E_1 \text{ and } u\in[b,3/2],\\ \dfrac{15-3u^2}{3u^3-18u^2+27u-4} \text{ for } P=L_{14}\cap E_1 \text{ and } u\in[3/2,2].\end{cases}$$

where a is a root of $3u^3-9u^2+3u+5$ on $[1,2]$, b is a root of $8u^3-24u^2+12u+7$ on $[1,3/2]$. Note that $a\in[1.355,1.356]$, $b\in[1.261,1.262]$.

Proof. **Step 1.** Suppose $P\in E_2\cup E_4$. Without loss of generality we can assume that $P\in E_4$. In this case we set $\mathscr{C}=E_4$. Then $\tau(\mathscr{C})=3-u$. The Zariski Decomposition of the divisor $D-vE_4$ is given by:

$$P(v)=\begin{cases}-K_T-(u-1)C_2-vE_4 \text{ for } v\in[0,2-u],\\ -K_T-(u-1)C_2-vE_4-(u+v-2)E_5 \text{ for } v\in[2-u,1],\\ -K_T-(u-1)C_2-vE_4-(u+v-2)E_5-(v-1)(L_{14}+2L_{24}+E_2) \text{ for } v\in[1,3-u].\end{cases}$$

and

$$N(v)=\begin{cases}0 \text{ for } v\in[0,2-u],\\ (u+v-2)E_5 \text{ for } v\in[2-u,1],\\ (u+v-2)E_5+(v-1)(L_{14}+2L_{24}+E_2) \text{ for } v\in[1,3-u].\end{cases}$$

Moreover,

$$P(v)^2=\begin{cases}5-u^2-2v^2 \text{ for } v\in[0,2-u],\\ 9+2uv-4u-4v-v^2 \text{ for } v\in[2-u,1],\\ 2(2-v)(3-u-v) \text{ for } v\in[1,3-u].\end{cases}$$

and

$$P(v)\cdot\mathscr{C}=\begin{cases}2v \text{ for } v\in[0,2-u],\\ 2-u+v \text{ for } v\in[2-u,1],\\ 5-u-2v \text{ for } v\in[1,3-u].\end{cases}$$

Thus,

$$S_D(\mathscr{C}) = \frac{1}{5-u^2}\Big(\int_0^{2-u} 5-u^2-2v^2 dv + \int_{2-u}^{1} 9+2uv-4u-4v-v^2 dv + \\ + \int_1^{3-u} 2(2-v)(3-u-v)dv\Big) = \frac{16+3u-9u^2+2u^3}{15-3u^2}.$$

Thus, $\delta_P(T,D) \leq \frac{15-3u^2}{16+3u-9u^2+2u^3}$ for $P \in E_4$. Note that we have:

- if $P \in E_4 \backslash (E_5 \cup L_{14} \cup L_{24} \cup L_{34})$

$$h_D(v) = \begin{cases} 2v^2 \text{ for } v \in [0, 2-u], \\ \frac{(2-u+v)^2}{2} \text{ for } v \in [2-u, 1], \\ \frac{(5-u-2v)^2}{2} \text{ for } v \in [1, 3-u]. \end{cases}$$

- if $P = E_4 \cap E_5$

$$h_D(v) = \begin{cases} 2v^2 \text{ for } v \in [0, 2-u], \\ \frac{(2-u+v)(u+3v-2)}{2} \text{ for } v \in [2-u, 1], \\ \frac{(u+1)(5-u-2v)}{2} \text{ for } v \in [1, 3-u]. \end{cases}$$

- if $P \in E_4 \cap (L_{14} \cup L_{24})$

$$h_D(v) \leq \begin{cases} 2v^2 \text{ for } v \in [0, 2-u], \\ \frac{(2-u+v)^2}{2} \text{ for } v \in [2-u, 1], \\ \frac{(5-u-2v)(1-u+2v)}{2} \text{ for } v \in [1, 3-u]. \end{cases}$$

So we have

- if $P \in E_4 \backslash (E_5 \cup L_{14} \cup L_{24} \cup L_{34})$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P) = \frac{2}{5-u^2}\Big(\int_0^{2-u} 2v^2 dv + \int_{2-u}^{1} \frac{(2-u+v)^2}{2} dv + \int_1^{3-u} \frac{(5-u-2v)^2}{2} dv\Big) = \\ = \frac{9+6u-9u^2+2u^3}{15-3u^2} \leq \frac{16+3u-9u^2+2u^3}{15-3u^2}.$$

- if $P = E_4 \cap E_5$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P) = \frac{2}{5-u^2}\Big(\int_0^{2-u} 2v^2 dv + \int_{2-u}^{1} \frac{(2-u+v)(u+3v-2)}{2} dv + \\ + \int_1^{3-u} \frac{(u+1)(5-u-2v)}{2} dv\Big) = \frac{11-u^3}{15-3u^2}.$$

- if $P \in E_4 \cap (L_{14} \cup L_{24})$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P)=\frac{2}{5-u^2}\Big(\int_0^{2-u}2v^2dv+ \\ +\int_{2-u}^1\frac{(2-u+v)^2}{2}dv+\int_1^{3-u}\frac{(5-u-2v)(1-u+2v)}{2}dv\Big)= \\ =\frac{2u^3-6u^2+8}{15-3u^2}\leq\frac{16+3u-9u^2+2u^3}{15-3u^2}.$$

We obtain that

$$\delta_P(T,D)=\frac{15-3u^2}{16+3u-9u^2+2u^3}\text{ for }P\in E_4\backslash E_5\text{ and }u\in[1,2].$$

and

$$\delta_P(T,D)\geq\begin{cases}\dfrac{15-3u^2}{16+3u-9u^2+2u^3}\text{ for }P=E_4\cap E_5\text{ and }u\in[1,a],\\ \dfrac{15-3u^2}{11-u^3}\text{ for }P=E_4\cap E_5\text{ and }u\in[a,2].\end{cases}$$

where a is a root of $3u^3-9u^2+3u+5$ on $[1,2]$. Note that $a\in[1.355,1.356]$.

Step 2. Suppose $P\in E_3\cup E_5$. Without loss of generality we can assume that $P\in E_5$. In this case we set $\mathscr{C}=E_5$. Then $\tau(\mathscr{C})=2$. The Zariski Decomposition of the divisor $D-vE_5$ is given by:

$$P(v)=\begin{cases}-K_T-(u-1)C_2-vE_5-\frac{v}{2}E_4\text{ for }v\in[0,1],\\ -K_T-(u-1)C_2-vE_5-\frac{v}{2}E_4-(v-1)L_{45}\text{ for }v\in[1,u],\\ -K_T-(u-1)C_2-vE_5-\frac{v}{2}E_4-(v-1)L_{45}-(v-u)C_2\text{ for }v\in[u,2].\end{cases}$$

and

$$N(v)=\begin{cases}\frac{v}{2}E_4\text{ for }v\in[0,1],\\ \frac{v}{2}E_4+(v-1)L_{45}\text{ for }v\in[1,u],\\ \frac{v}{2}E_4+(v-1)L_{45}+(v-u)C_2\text{ for }v\in[u,2].\end{cases}$$

Moreover,

$$P(v)^2=\begin{cases}5-4v+2uv-u^2-\frac{v^2}{2}\text{ for }v\in[0,1],\\ 6-6v+\frac{v^2}{2}+2uv-u^2\text{ for }v\in[1,u],\\ \frac{3(2-v)^2}{2}\text{ for }v\in[u,2].\end{cases}$$

and

$$P(v)\cdot\mathscr{C}=\begin{cases}2-u+v/2\text{ for }v\in[0,1],\\ 3-u-v/2\text{ for }v\in[1,u],\\ 3-3v/2\text{ for }v\in[u,2].\end{cases}$$

Thus,

$$S_D(\mathscr{C}) = \frac{1}{5-u^2}\Big(\int_0^1 5-4v+2uv-u^2-\frac{v^2}{2}dv+\int_1^u 6-6v+\frac{v^2}{2}+2uv-u^2dv+ \\ +\int_u^2 \frac{3(2-v)^2}{3}dv\Big) = \frac{11-u^3}{15-3u^2}.$$

Thus, $\delta_P(T,D) \leq \frac{15-3u^2}{11-u^3}$ for $P \in E_5$. Note that we have:

- if $P \in E_5 \backslash (E_4 \cup C_2 \cup L_{45})$ then

$$h_D(v) = \begin{cases} \frac{(2-u+v/2)^2}{2} \text{ for } v \in [0,1], \\ \frac{(3-u-v/2)^2}{2} \text{ for } v \in [1,u], \\ \frac{(3-3v/2)^2}{2} \text{ for } v \in [u,2]. \end{cases}$$

- if $P = E_5 \cap C_2$ then

$$h_D(v) = \begin{cases} \frac{(2-u+v/2)^2}{2} \text{ for } v \in [0,1], \\ \frac{(3-u-v/2)^2}{2} \text{ for } v \in [1,u], \\ \frac{3(2-v)(6-4u+v)}{8} \text{ for } v \in [u,2]. \end{cases}$$

- if $P = E_5 \cap L_{45}$ then

$$h_D(v) = \begin{cases} \frac{(2-u+v/2)^2}{2} \text{ for } v \in [0,1], \\ \frac{(6-2u-v)(2-2u+3v)}{8} \text{ for } v \in [1,u], \\ \frac{3(2-v)(v+2)}{8} \text{ for } v \in [u,2]. \end{cases}$$

So we have

- if $P \in E_5 \backslash (E_4 \cup C_2 \cup L_{45})$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P) = \frac{2}{5-u^2}\Big(\int_0^1 \frac{(2-u+v/2)^2}{2}dv+\int_1^u \frac{(3-u-v/2)^2}{2}dv+ \\ +\int_u^2 \frac{(3-3v/2)^2}{2}dv\Big) = \frac{21+6u-18u^2+5u^3}{2(15-3u^2)} \leq \frac{11-u^3}{15-3u^2}.$$

- if $P = E_5 \cap C_2$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P) = \frac{2}{5-u^2}\Big(\int_0^1 \frac{(2-u+v/2)^2}{2}dv+\int_1^u \frac{(3-u-v/2)^2}{2}dv+ \\ +\int_u^2 \frac{3(2-v)(6-4u+v)}{8}dv\Big) = \frac{45-30u+2u^3}{2(15-3u^2)} \leq \frac{11-u^3}{15-3u^2}.$$

- if $P = E_5 \cap L_{45}$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P) = \frac{2}{5-u^2}\Big(\int_0^1 \frac{(2-u+v/2)^2}{2}dv + \int_1^u \frac{(6-2u-v)(2-2u+3v)}{8}dv + \\ + \frac{3(2-v)(v+2)}{8}dv\Big) = \frac{26-12u^2+3u^3}{2(15-3u^2)} \leq \frac{11-u^3}{15-3u^2}.$$

We obtain that

$$\delta_P(T,D) = \frac{15-3u^2}{11-u^3} \text{ for } P \in (E_3 \cup E_5)\backslash(E_2 \cup E_4) \text{ and } u \in [1,2].$$

Step 3. Suppose $P \in L_{24}$. In this case we set $\mathscr{C} = L_{24}$. Then $\tau(\mathscr{C}) = 3-u$. The Zariski Decomposition of the divisor $D - vL_{24}$ is given by:

$$P(v) = \begin{cases} D - vL_{24} - \frac{v}{2}(E_2+E_4) \text{ for } v \in [0,4-2u], \\ D - vL_{24} - (u+v-2)(E_2+E_4) - (2u+v-4)(E_3+E_5) \text{ for } v \in [4-2u,3-u]. \end{cases}$$

and

$$N(v) = \begin{cases} \frac{v}{2}(E_2+E_4) \text{ for } v \in [0,4-2u], \\ (u+v-2)(E_2+E_4) + (2u+v-4)(E_3+E_5) \text{ for } v \in [4-2u,3-u]. \end{cases}$$

Moreover,

$$P(v)^2 = \begin{cases} -u^2-2v+5 \text{ for } v \in [0,4-2u], \\ (u+v-3)(3u+v-7) \text{ for } v \in [4-2u,3-v]. \end{cases}$$

and

$$P(v)\cdot\mathscr{C} = \begin{cases} 1 \text{ for } v \in [0,4-2u], \\ 5-2u-v \text{ for } v \in [4-2u,3-u]. \end{cases}$$

Thus,

$$S_D(\mathscr{C}) = \frac{1}{5-u^2}\Big(\int_0^{4-2u} -u^2-2v+5dv + \int_{4-2u}^{3-uu} (u+v-3)(3u+v-7)dv\Big) = \\ = \frac{4u^3-15u^2+6u+17}{15-3u^2}.$$

Thus, $\delta_P(T,D) \leq \frac{15-3u^2}{4u^3-15u^2+6u+17}$ for $P \in L_{24}$. If $P \in L_{24}\backslash(E_2 \cup E_4)$ then

$$h_D(v) = \begin{cases} \frac{1}{2} \text{ for } v \in [0,4-2u], \\ \frac{(5-2u-v)^2}{2} \text{ for } v \in [4-2u,3-u]. \end{cases}$$

So for $P \in L_{24}\backslash(E_2 \cup E_4)$ we have

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P) = \frac{2}{5-u^2}\Big(\int_0^{4-2u}\frac{1}{2}dv + \int_{4-2u}^{3-u}\frac{(5-2u-v)^2}{2}dv = \\ = \frac{u^3-6u^2+6u+5}{15-3u^2} \le \frac{4u^3-15u^2+6u+17}{15-3u^2}.$$

We obtain that

$$\delta_P(T,D) = \frac{15-3u^2}{u^3-6u^2+6u+5} \text{ for } P \in L_{24}\backslash(E_2\cup E_4) \text{ and } u \in [1,2].$$

Step 4.1. Suppose $P \in L_{12} \cup L_{14}$ and $u \in [1,3/2]$. Without loss of generality, we can assume that $P \in L_{14}$. In this case we set $\mathscr{C} = L_{14}$. Then $\tau(\mathscr{C}) = 3-u$. The Zariski Decomposition of the divisor $D - vL_{14}$ is given by:

$$P(v) = \begin{cases} D - vL_{14} - \frac{v}{2}E_4 \text{ for } v \in [0,2-u], \\ D - vL_{14} - \frac{v}{2}E_4 - (u+v-2)E_1 \text{ for } v \in [2-u,1], \\ D - vL_{14} - \frac{v}{2}E_4 - (u+v-2)E_1 - (v-1)L_{23} \text{ for } v \in [1,4-2u], \\ D - vL_{14} - (u+v-2)(E_1+E_4) - (v-1)L_{23} - (2u+v-4)E_5 \text{ for } v \in [4-2u,3-u]. \end{cases}$$

and

$$N(v) = \begin{cases} \frac{v}{2}E_4 \text{ for } v \in [0,2-u], \\ \frac{v}{2}E_4 + (u+v-2)E_1 \text{ for } v \in [2-u,1], \\ \frac{v}{2}E_4 + (u+v-2)E_1 + (v-1)L_{23} \text{ for } v \in [1,4-2u], \\ (u+v-2)(E_1+E_4) + (v-1)L_{23} + (2u+v-4)E_5 \text{ for } v \in [4-2u,3-u]. \end{cases}$$

Moreover

$$P(v)^2 = \begin{cases} 5 - 2v - \frac{v^2}{2} - u^2 \text{ for } v \in [0,2-u], \\ 9 - 4u - 6v + \frac{v^2}{2} + 2uv \text{ for } v \in [2-u,1], \\ \frac{(v-2)(3v+4u-10)}{2} \text{ for } v \in [1,4-2u], \\ 2(u+v-3)^2 \text{ for } v \in [4-2u,3-u]. \end{cases}$$

and

$$P(v)\cdot\mathscr{C} = \begin{cases} v/2+1 \text{ for } v \in [0,2-u], \\ 3-u-v/2 \text{ for } v \in [2-u,1], \\ 4-u-3v/2 \text{ for } v \in [1,4-2u], \\ 2(3-u-v) \text{ for } v \in [4-2u,3-u]. \end{cases}$$

Thus,

$$S_D(\mathscr{C}) = \frac{1}{5-u^2}\Big(\int_0^{2-u} 5-2v-\frac{v^2}{2}-u^2 dv + \int_{2-u}^1 9-4u-6v+\frac{v^2}{2}+2uvdv+ \\ +\int_1^{4-2u} \frac{(v-2)(3v+4u-10)}{2}dv + \int_{4-2u}^{3-u} 2(u+v-3)^2 dv\Big) = \\ = \frac{3u^3-12u^2+6u+13}{15-3u^2}.$$

Thus, $\delta_P(T,D) \leq \frac{15-3u^2}{3u^3-12u^2+6u+13}$ for $P \in L_{14}$. Note that we have:

- if $P \in L_{14} \backslash (E_4 \cup E_1 \cup L_{23} \cup E_5)$ then

$$h_D(v) = \begin{cases} \frac{(v/2+1)^2}{2} \text{ for } v \in [0, 2-u], \\ \frac{(3-u-v/2)^2}{2} \text{ for } v \in [2-u, 1], \\ \frac{(4-u-3v/2)^2}{2} \text{ for } v \in [1, 4-2u], \\ 2(3-u-v)^2 \text{ for } v \in [4-2u, 3-u]. \end{cases}$$

- if $P = L_{14} \cap E_1$ then

$$h_D(v) = \begin{cases} \frac{(v/2+1)^2}{2} \text{ for } v \in [0, 2-u], \\ \frac{(6-2u-v)(2u+3v-2)}{8} \text{ for } v \in [2-u, 1], \\ \frac{(8-2u-3v)(2u+v)}{8} \text{ for } v \in [1, 4-2u], \\ (3-u-v) \text{ for } v \in [4-2u, 3-u]. \end{cases}$$

- if $P = L_{14} \cap L_{23}$ then

$$h_D(v) = \begin{cases} \frac{(v/2+1)^2}{2} \text{ for } v \in [0, 2-u], \\ \frac{(3-u-v/2)^2}{2} \text{ for } v \in [2-u, 1], \\ \frac{(8-2u-3v)(4-2u+v)}{8} \text{ for } v \in [1, 4-2u], \\ 2(2-u)(3-u-v) \text{ for } v \in [4-2u, 3-u]. \end{cases}$$

So we have

- if $P \in L_{14} \backslash (E_4 \cup E_1 \cup L_{23} \cup E_5)$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^{2-u} \frac{(v/2+1)^2}{2}dv + \int_{2-u}^1 \frac{(3-u-v/2)^2}{2}dv+ \\ +\int_1^{4-2u} \frac{(4-u-3v/2)^2}{2}dv + \int_{4-2u}^{3-u} 2(3-u-v)^2 dv\Big) = \\ = \frac{21-u^3-6u}{2(15-3u^2)} \leq \frac{3u^3-12u^2+6u+13}{15-3u^2}.$$

- if $P = L_{14} \cap E_1$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P) = \frac{2}{5-u^2}\Big(\int_0^{2-u}\frac{(v/2+1)^2}{2}dv + \int_{2-u}^{1}\frac{(6-2u-v)(2u+3v-2)}{8}dv + \\ + \int_1^{4-2u}\frac{(8-2u-3v)(2u+v)}{8}dv + \int_{4-2u}^{3-u}(3-u-v)dv\Big) = \\ = \frac{19-2u^3}{2(15-3u^2)}.$$

- if $P = L_{14} \cap L_{23}$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P) = \frac{2}{5-u^2}\Big(\int_0^{2-u}\frac{(v/2+1)^2}{2}dv + \int_{2-u}^{1}\frac{(3-u-v/2)^2}{2}dv + \\ + \int_1^{4-2u}\frac{(8-2u-3v)(4-2u+v)}{8}dv + \int_{4-2u}^{3-u}2(2-u)(3-u-v)dv\Big) = \\ = \frac{26-12u^2+3u^3}{2(15-3u^2)} \leq \frac{3u^3-12u^2+6u+13}{15-3u^2}.$$

We obtain that

$$\delta_P(T,D) = \frac{15-3u^2}{3u^3-12u^2+6u+13} \text{ for } P \in L_{14}\backslash(E_4 \cup E_5 \cup E_1) \text{ and } u \in [1,3/2]$$

and

$$\delta_P(T,D) \geq \begin{cases} \dfrac{15-3u^2}{3u^3-12u^2+6u+13} \text{ for } P = L_{14} \cap E_1 \text{ and } u \in [1,b], \\ \dfrac{2(15-3u^2)}{19-2u^3} \text{ for } P = L_{14} \cap E_1 \text{ and } u \in [b,3/2] \end{cases}$$

where b is a root of $8u^3 - 24u^2 + 12u + 7$ on $[1,3/2]$. Note that $b \in [1.261, 1.262]$.

Step 4.2. Suppose $P \in L_{12} \cup L_{14}$ and $u \in [3/2,2]$. Without loss of generality, we can assume that $P \in L_{14}$. In this case we set $\mathscr{C} = L_{14}$. Then $\tau(\mathscr{C}) = 3-u$. The Zariski Decomposition of the divisor $D - vL_{14}$ is given by:

$$P(v) = \begin{cases} D - vL_{14} - \frac{v}{2}E_4 \text{ for } v \in [0,2-u], \\ D - vL_{14} - \frac{v}{2}E_4 - (u+v-2)E_1 \text{ for } v \in [2-u,4-2u], \\ D - vL_{14} - (u+v-2)(E_1+E_4) - (2u+v-4)E_5 \text{ for } v \in [4-2u,1], \\ D - vL_{14} - (u+v-2)(E_1+E_4) - (v-1)L_{23} - (2u+v-4)E_5 \text{ for } v \in [1,3-u]. \end{cases}$$

and

$$N(v)=\begin{cases}\frac{v}{2}E_4 \text{ for } v\in[0,2-u],\\ \frac{v}{2}E_4+(u+v-2)E_1 \text{ for } v\in[2-u,4-2u],\\ (u+v-2)(E_1+E_4)+(2u+v-4)E_5 \text{ for } v\in[4-2u,1],\\ (u+v-2)(E_1+E_4)+(v-1)L_{23}+(2u+v-4)E_5 \text{ for } v\in[1,3-u].\end{cases}$$

Moreover

$$P(v)^2=\begin{cases}5-2v-\frac{v^2}{2}-u^2 \text{ for } v\in[0,2-u],\\ 9-4u-6v+\frac{v^2}{2}+2uv \text{ for } v\in[2-u,4-2u],\\ 2u^2+4uv+v^2-12u-10v+17 \text{ for } v\in[4-2u,1],\\ 2(u+v-3)^2 \text{ for } v\in[1,3-u].\end{cases}$$

and

$$P(v)\cdot\mathscr{C}=\begin{cases}1+v/2 \text{ for } v\in[0,2-u],\\ 3-u-v/2 \text{ for } v\in[2-u,4-2u],\\ 5-2u-v \text{ for } v\in[4-2u,1],\\ 2(3-u-v) \text{ for } v\in[1,3-u].\end{cases}$$

Thus,

$$S_D(\mathscr{C})=\frac{1}{5-u^2}\Big(\int_0^{2-u}5-2v-\frac{v^2}{2}-u^2dv+\int_{2-u}^{4-2u}9-4u-6v+\frac{v^2}{2}+2uvdv+ \\ +\int_{4-2u}^{1}2u^2+4uv+v^2-12u-10v+17dv+ \\ +\int_1^{3-u}2(u+v-3)^2dv\Big)=\frac{3u^3-12u^2+6u+13}{15-3u^2}.$$

Thus, $\delta_P(T,D)\le\frac{15-3u^2}{3u^3-12u^2+6u+13}$ for $P\in L_{14}$. Note that we have:

- if $P\in L_{14}\backslash(E_4\cup E_1\cup L_{23}\cup E_5)$ then

$$h_D(v)=\begin{cases}\frac{(1+v/2)^2}{2} \text{ for } v\in[0,2-u],\\ \frac{(3-u-v/2)^2}{2} \text{ for } v\in[2-u,4-2u],\\ \frac{(5-2u-v)^2}{2} \text{ for } v\in[4-2u,1],\\ 2(3-u-v)^2 \text{ for } v\in[1,3-u].\end{cases}$$

- if $P = L_{14} \cap E_1$ then

$$h_D(v) = \begin{cases} \frac{(1+v/2)^2}{2} \text{ for } v \in [0, 2-u], \\ \frac{(6-2u-v)(2u+3v-2)}{8} \text{ for } v \in [2-u, 4-2u], \\ \frac{(v+1)(5-2u-v)}{2} \text{ for } v \in [4-2u, 1], \\ 2(3-u-v) \text{ for } v \in [1, 3-u]. \end{cases}$$

- if $P = L_{14} \cap L_{23}$ then

$$h_D(v) = \begin{cases} \frac{(1+v/2)^2}{2} \text{ for } v \in [0, 2-u], \\ \frac{(3-u-v/2)^2}{2} \text{ for } v \in [2-u, 4-2u], \\ \frac{(5-2u-v)^2}{2} \text{ for } v \in [4-2u, 1], \\ 2(2-u)(3-u-v) \text{ for } v \in [1, 3-u]. \end{cases}$$

- if $P \in L_{14} \backslash (E_4 \cup E_1 \cup L_{23} \cup E_5)$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2} \Big(\int_0^{2-u} \frac{(v/2+1)^2}{2} dv + \int_{2-u}^{4-2u} \frac{(3-u-v/2)^2}{2} dv + \\ + \int_{4-2u}^{1} \frac{(5-2u-v)^2}{2} dv + \int_1^{3-u} 2(3-u-v)^2 dv \Big) = \\ = \frac{7u^3 - 36u^2 + 48u - 6}{2(15-3u^2)} \leq \frac{3u^3 - 12u^2 + 6u + 13}{15-3u^2}.$$

- if $P = L_{14} \cap E_1$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2} \Big(\int_0^{2-u} \frac{(v/2+1)^2}{2} dv + \int_{2-u}^{4-2u} \frac{(6-2u-v)(2u+3v-2)}{8} dv + \\ + \int_{4-2u}^{1} \frac{(v+1)(5-2u-v)}{2} dv + \int_1^{3-u} 2(3-u-v) dv \Big) = \\ = \frac{3u^3 - 18u^2 + 27u - 4}{15-3u^2}.$$

- if $P = L_{14} \cap L_{23}$ then

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2} \Big(\int_0^{2-u} \frac{(v/2+1)^2}{2} dv + \int_{2-u}^{4-2u} \frac{(3-u-v/2)^2}{2} dv + \\ + \int_{4-2u}^{1} \frac{(5-2u-v)^2}{2} dv + \int_1^{3-u} 2(2-u)(3-u-v) dv \Big) = \\ = \frac{3u^3 - 12u^2 + 26}{2(15-3u^2)} \leq \frac{3u^3 - 12u^2 + 6u + 13}{15-3u^2}.$$

We obtain that

$$\delta_P(T,D) = \frac{15-3u^2}{3u^3-12u^2+6u+13} \text{ for } P \in L_{14} \backslash (E_1 \cup E_4 \cup E_5) \text{ and } u \in [3/2,2].$$

and

$$\delta_P(T,D) \geq \frac{15-3u^2}{3u^3-18u^2+27u-4} \text{ for } P = L_{14} \cap E_1 \text{ and } u \in [3/2,2].$$

□

Corollary 12.3.5. *Let P be a point in T that is contained in $L_{12} \cup L_{14} \cup L_{24} \cup E_2 \cup E_3 \cup E_4 \cup E_4$ then*

$$\delta_P(T,D) \geq \begin{cases} \dfrac{15-3u^2}{16+3u-9u^2+2u^3} \text{ for } u \in [1,a], \\ \dfrac{15-3u^2}{11-u^3} \text{ for } u \in [a,2]. \end{cases}$$

Corollary 12.3.6. *Suppose O is a point on a del Pezzo surface $\overline{T}$ with two $\mathbb{A}_1$ singularities and nine lines such that $\delta_O(T) \leq \frac{6}{5}$ then*

$$\delta_O(\overline{T},\overline{D}) \geq \begin{cases} \dfrac{15-3u^2}{16+3u-9u^2+2u^3} \text{ for } u \in [1,a], \\ \dfrac{15-3u^2}{11-u^3} \text{ for } u \in [a,2]. \end{cases}$$

12.3.3 Du Val Del Pezzo surface of degree 4 with $\mathbb{A}_2$ singularity

Now, let us use the notations and assumptions of Section 2 with a minor difference: we assume that $\overline{T}$ has a singular point of type $\mathbb{A}_2$. Let us show that in the case when O is the singular point of the surface $\overline{T}$ we have

$$\delta_O(\overline{T},\overline{D}) = \frac{15-3u^2}{u^3-6u^2+19},$$

which immediately implies that $\delta_O(\overline{T}, W^{\overline{T}}_{\bullet,\bullet}) \leq \frac{80}{83}$. In this case, the morphism η is a blow up of $\mathbb{P}^2$ at points P_1, P_2, and P_3 in general position; after that blowing up a point P_4 which belongs to the exceptional divisor corresponding to P_3 and no other negative curve and after that a point P_5 which belongs to the exceptional divisor corresponding to P_4 and no other negative curve. In Part I we proved:

$$\delta_P(T) = \begin{cases} 6/7 \text{ if } P \in E_3 \cup E_4, \\ 8/7 \text{ if } P \in (L_{13} \cup L_{23} \cup L_{34} \cup E_5) \backslash (E_3 \cup E_4), \\ 4/3 \text{ if } P \in (L_{12} \cup C_2) \cap (E_1 \cup E_2), \\ 18/13 \text{ if } P \in (L_{12} \cup C_2 \cup E_1 \cup E_2) \backslash ((L_{12} \cup C_2) \cap (E_1 \cup E_2)), \\ 3/2 \text{ otherwise.} \end{cases}$$

where E_1, E_2, E_3, E_4, E_5 are exceptional divisors corresponding to P_1, P_2, P_3, P_4, P_5 respectively, C_2 is a strict transform of a (-1)-curve coming from the conic on $\mathbb{P}^2$, L_{ij} are strict transforms of the lines passing through P_i and P_j for $(i,j) \in \{(1,2),(1,3),(2,3)\}$ and L_{34} is a strict transform of a (-1)-curve coming from a line passing through P_3 on $\mathbb{P}^2$. The dual graph of (-1) and (-2)-curves is given in the following picture: Now let's prove that:

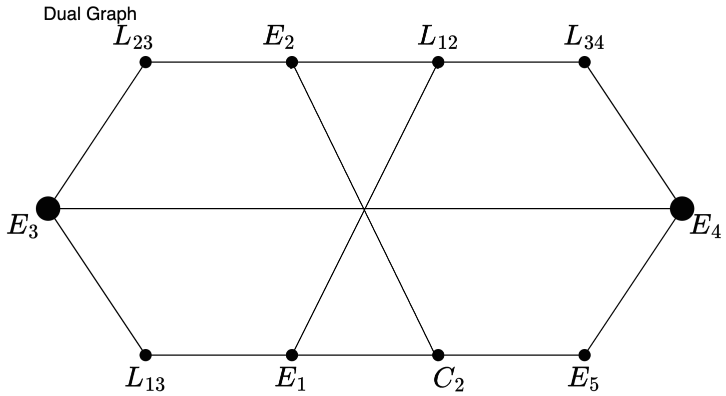


Figure 12.3: Polarized δ-invariants: $\mathbb{A}_2$ singularity

Lemma 12.3.7. *Suppose P is a point on T and $D = -K_T - (u-1)C_2$ with $D^2 = 5 - u^2$ then*

$$\delta_P(T,D) = \frac{15-3u^2}{u^3-6u^2+19} \text{ for } P \in E_4 \backslash (L_{34} \cup E_5).$$

Proof. Suppose $P \in E_4 \backslash (L_{34} \cup E_5)$. In this case we set $\mathscr{C} = E_4$. Then $\tau(\mathscr{C}) = 2$. The Zariski Decomposition of the divisor $D - vE_4$ is given by:

$$P(v) = \begin{cases} -K_T - (u-1)C_2 - vE_4 - \frac{v}{2}E_3 \text{ for } v \in [0, 2-u], \\ -K_T - (u-1)C_2 - vE_4 - \frac{v}{2}E_3 - (u+v-2)E_5 \text{ for } v \in [2-u, 1], \\ -K_T - (u-1)C_2 - vE_4 - \frac{v}{2}E_3 - (u+v-2)E_5 - (v-1)L_{34} \text{ for } v \in [1,2]. \end{cases}$$

and

$$N(v) = \begin{cases} \frac{v}{2}E_3 \text{ for } v \in [0, 2-u], \\ \frac{v}{2}E_3 + (u+v-2)E_5 \text{ for } v \in [2-u, 1], \\ \frac{v}{2}E_3 + (u+v-2)E_5 + (v-1)L_{34} \text{ for } v \in [1,2]. \end{cases}$$

Moreover,

$$P(v)^2 = \begin{cases} 5 - u^2 - \frac{3v^2}{2} \text{ for } v \in [0, 2-u], \\ 9 - 4u - 4v + 2uv - 1/2v^2 \text{ for } v \in [2-u, 1], \\ \frac{(v-2)(v+4u-10)}{2} \text{ for } v \in [1,2]. \end{cases}$$

and

$$P(v)\cdot\mathscr{C} = \begin{cases} 3v/2 \text{ for } v\in[0,2-u], \\ 2-u+v/2 \text{ for } v\in[2-u,1], \\ 3-u-v/2 \text{ for } v\in[1,2]. \end{cases}$$

Thus,

$$S_D(\mathscr{C}) = \frac{1}{5-u^2}\Big(\int_0^{2-u} 5-u^2-\frac{3v^2}{2}dv+\int_{2-u}^{1} 9-4u-4v+2uv-1/2v^2dv+ \\ +\int_1^2 \frac{(v-2)(v+4u-10)}{2}dv\Big) = \frac{19+u^3-6u^2}{15-3u^2}$$

Thus, $\delta_P(T,D)\leq\frac{15-3u^2}{19+u^3-6u^2}$ for $P\in E_4$. Note that for $P\in E_4\backslash(E_5\cup L_{34})$ we have:

$$h_D(v) = \begin{cases} \frac{9v^2}{8} \text{ for } v\in[0,2-u], \\ \frac{(2-u+v/2)^2}{2} \text{ for } v\in[2-u,1], \\ \frac{(3-u-v/2)^2}{2} \text{ for } v\in[1,2]. \end{cases}$$

So we have

$$S_D(W^{\mathscr{C}}_{\bullet,\bullet};P) = \frac{2}{5-u^2}\Big(\int_0^{2-u}\frac{9v^2}{8}dv+\int_{2-u}^{1}\frac{(2-u+v/2)^2}{2}dv+\int_1^2\frac{(3-u-v/2)^2}{2}dv\Big) = \\ = \frac{21+6u-18u^2+5u^3}{2(15-3u^2)} \leq \frac{19+u^3-6u^2}{15-3u^2}.$$

So we obtain that

$$\delta_P(T,D) = \frac{15-3u^2}{u^3-6u^2+19} \text{ for } P\in E_4\backslash(L_{34}\cup E_5).$$

□

Corollary 12.3.8. *We have $S(W^{\overline{T}}_{\bullet,\bullet};E_4)=\frac{83}{80}$, which implies that $\delta_O(\overline{T},W^{\overline{T}}_{\bullet,\bullet})\leq\frac{80}{83}$.*

Proof.

$$S\big(W^{\overline{T}}_{\bullet,\bullet};E_4\big) = \\ = \frac{3}{(-K_X)^3}\int_1^2\int_0^\infty \mathrm{vol}\big(P(u)\big|_{\overline{T}}-vE_4\big)dvdu+\frac{3}{(-K_X)^3}\int_0^1\int_0^\infty \mathrm{vol}\big(P(u)\big|_{\overline{T}}-vE_4\big)dvdu = \\ = \frac{3}{20}\int_1^2(5-u^2)S_D(E_4)du+\frac{3}{5} = \frac{3}{20}\int_1^2(5-u^2)\frac{19+u^3-6u^2}{15-3u^2}du+\frac{3}{5} = \frac{83}{80}.$$

□

Bibliography

Abban, H., & Zhuang, Z. (2022). K-stability of Fano varieties via admissible flags. *Forum of Mathematics, Pi*, *10 e15*, 43 pp. doi: 10.1017/fmp.2022.11

Araujo, C., Castravet, A.-M., Cheltsov, I., Fujita, K., Kaloghiros, A.-S., Martinez-Garcia, J., ... Viswanathan, N. (2023). The Calabi problem for Fano threefolds. *Cambridge University Press*. doi: 10.1017/9781009193382

Ascher, K., DeVleming, K., & Liu. (2023). K-stability and birational models of moduli of quartic K3 surfaces. *Invent. math.*, *232*, 471–552 pp. doi: 10.1007/s00222-022-01170-5

Aubin, T. (1978). Équations du type monge-ampère sur les variétés Kähleriennes compactes. *Bull. Sci. Math. Sér. 2*, *102*, 63-95 pp.

Belousov, G., & Loginov, K. (2024). K-stability of Fano threefolds of rank 3 and degree 14. *Ann Univ Ferrara*, *70*, 1093–1114 pp. doi: 10.1007/s11565-024-00526-4

Blum, H., & Jonsson, M. (2020). Thresholds, valuations, and K-stability. *Adv. Math.*, *365*, 107062, 57. doi: 10.1016/j.aim.2020.107062

Blum, H., & Xu, C. (2019). Uniqueness of K-polystable degenerations of Fano varieties. *Ann. Math.*, *190*, 609–656 pp. doi: 10.4007/annals.2019.190.2.4

Cheltsov, I. (2009). On singular cubic surfaces. *Asian Journal of Mathematics*, *13 (2)*, 191-214 pp.

Cheltsov, I. (2024). K-stability of Fano 3-folds of picard rank 3 and degree 22. , 22 pp. doi: arXiv:2401.02818

Cheltsov, I., Denisova, E., & Fujita, K. (2024). K-stable smooth Fano threefolds of picard rank two. *Forum of Mathematics, Sigma.*, *12 (e41)*, 64 pp. doi: 10.1017/fms.2024.5

Cheltsov, I., Fujita, K., Kishimoto, T., & Okada, T. (2023). K-stable divisors in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of degree $(1,1,2)$. *Nagoya Mathematical Journal*, *251*, 686-714 pp. doi: 10.1017/nmj.2023.5

Cheltsov, I., & Kosta, D. (2014). Computing α-invariants of singular del Pezzo surfaces. *Journal of Geometric Analysis*, *24*, 798-842 pp. doi: 10.1007/s12220-012-9357-6

Cheltsov, I., & Prokhorov, Y. (2021). Del Pezzo surfaces with infinite automorphism groups. *Algebraic Geometry*, *8 (3)*, 319–357 pp. doi: 10.14231/AG-2021-008

Cheltsov, I., Przyjalkowski, V., & Shramov, C. (2019). Fano threefolds with infinite automorphism groups. *Izvestia: Mathematics*, *83*, 860-907 pp.

Chen, X.-X., Donaldson, S., & Sun, S. (2015). Kähler-Einstein metrics on Fano manifolds. i, ii, iii. *J. Am.Math. Soc.*, *28 (1)*, 183–197, 199–234, 235–278 pp.

Coray, D., & Tsfasman, M. (1988). Arithmetic on singular del Pezzo surfaces. *Proceedings of the London Mathematical Society*, *s3-57 (1)*, 25-87 pp. doi: 10.1112/plms/s3-57.1.25

Denisova, E. (2023-24). δ-invariants of Du Val del Pezzo surfaces. *preprint*. doi: arXiv: 2410.19853,arXiv:2410.12512,arXiv:2311.14181,arXiv:2304.11412

Denisova, E. (2024a). K-stability of Fano 3-folds of picard rank 3 and degree 20. *Ann Univ Ferrara*, *70*, 987–1022 pp. doi: 10.1007/s11565-024-00516-6

Denisova, E. (2024b). On K-stability of $\mathbb{P}^3$ blown up along the disjoint union of a twisted cubic curve and a line. *Journal of the London Mathematical Society*, *109 (5), e12911*, 26 pp. doi: 10.1112/jlms.12911

Dervan, R. (2016). On K-stability of finite covers. *Bull. Lond. Math. Soc.*, *48 (4)*, 717-728 pp. doi: 10.1112/blms/bdw029

Dolgachev, I. V. (2012). *Classical Algebraic Geometry: A Modern View*. Cambridge, UK: Cambridge University Press. doi: 10.1017/CBO9781139084437

Donaldson, S. K. (1985). Anti Self-Dual Yang-Mills Connections over Complex Algebraic Surfaces and Stable Vector Bundles. *Proceedings of the London Mathematical Society*, 1–26 pp. doi: 10.1112/plms/s3-50.1.1.

Donaldson, S. K. (2002). Scalar curvature and stability of toric varieties. *Journal of Differential Geometry*, *62 (2)*, 289–349 pp. doi: 10.4310/jdg/1090950195

Fujita, K. (2016). On K-stability and the volume functions of $\mathbb{Q}$-Fano varieties. *Proc. Lond. Math. Soc.*, *113 (5)*, 541–582 pp. doi: 10.1112/plms/pdw037

Fujita, K. (2019). A valuative criterion for uniform K-stability of $\mathbb{Q}$-Fano varieties. *J. Reine Angew. Math.*, *751*, 309–338 pp. doi: 10.1515/crelle-2016-0055

Fujita, K., & Odaka, Y. (2018). On the K-stability of Fano varieties and anticanonical divisors. *Tohoku Math. J.*, *70*, 511–521 pp.

Ghigi, A., & Kollár, J. (2007). Kähler–Einstein metrics on orbifolds and Einstein metrics on spheres. *Comment. Math. Helv.*, *82*, 877–902 pp.

Iskovskikh, V. (1997). Fano 3-folds i. *Math. USSR, Izv.*, *11*, 485–527 pp.

Iskovskikh, V. (1998). Fano 3-folds ii. *Math. USSR, Izv.*, *12*, 469–506 pp.

Kosta, D. (2009). Del pezzo surfaces with Du Val singularities. *arXiv preprint arXiv:0904.0943*.

Li, C. (2017). K-semistability is equivariant volume minimization. *Duke Math. Jour.*, *166 (16)*, 3147–3218 pp. doi: 10.1215/00127094-2017-0026

Li, C., Tian, G., & Wang, F. (2021). On the Yau-Tian-Donaldson conjecture for singular Fano varieties. *Comm. Pure Appl. Math.*, *78 (8)*, 1748-1800 pp.

Li, C., Tian, G., & Wang, F. (2022). The uniform version of Yau-Tian-Donaldson conjecture for singular Fano varieties. *Peking Math. J.*, *5 (2)*, 383–426 pp.

Li, C., Wang, X., & Xu, C. (2019). On the proper moduli spaces of smoothable Kähler-Einstein Fano varieties. *Duke Math. J.*, *168 (8)*, 1387-1459 pp.

Liu, Y., Xu, C., & Zhuang, Z. (2022). Finite generation for valuations computing stability thresholds and applications to K-stability. *Annals of Mathematics*, *196*(2), 507–566.

Mabuchi, T., & Mukai, S. (1993). Stability and Einstein–Kähler metric of a quartic del Pezzo surface. In (Vol. 145, pp. 133–160 pp.). Lecture Notes in Pure and Applied Mathematics.

Matsushima, Y. (1957). Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kählérienne. *Nagoya Math. J.*, *11*, 145–150.

Mori, S., & Mukai, S. (1981). Classification of Fano 3-folds with $b_2 \geqslant 2$. *Manuscr. Math.*, *36*, 147–162 pp.

Mori, S., & Mukai, S. (2003). Classification of Fano 3-folds with $b_2 \geqslant 2$. *Manuscr. Math.*, *407*.

Odaka, Y. (2013). A generalization of the Ross-Thomas slope theory. *Osaka J. Math.*, *50*(1), 171–185 pp.

Odaka, Y., Spotti, C., & Sun, S. (2016). Compact moduli spaces of del Pezzo surfaces and Kähler–Einstein metrics. *J. Differ. Geom.*, *102 (1)*, 127-172 pp. doi: 10.4310/jdg/1452002879

Park, J., & Won, J. (2010a). Log-canonical thresholds on del Pezzo surfaces of degrees ≥ 2. *Nagoya Math. J.*, *200*, 1-26 pp. doi: 10.1215/00277630-2010-009

Park, J., & Won, J. (2010b). Log canonical thresholds on Gorenstein canonical del Pezzo surfaces. *Proc. Edinb. Math. Soc.*, *54 (1)*, 187-219 pp. doi: 10.1017/S001309150900039X

Reid, M. (1987). Young person's guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)* (Vol. 46, pp. 345–414). Amer. Math. Soc., Providence, RI.

Shah, J. (1981). Degenerations of K3 surfaces of degree 4. *Trans. Am. Math. Soc.*, *263 (2)*, 271–308 pp.

Spotti, C., Sun, S., & Yao, C.-J. (2016). Existence and deformations of Kähler-Einstein metrics on smoothable $\mathbb{Q}$-Fano varieties. *Duke Math. J.*, *165 (16)*, 3043-3083 pp.

Tian, G. (1990). On Calabi's conjecture for complex surfaces with positive first chern class. *Invent. Math.*, *101*, 101–172 pp.

Tian, G. (1997). Kähler–Einstein metrics with positive scalar curvature. *Invent math*, *130*, 1–37 pp. doi: 10.1007/s002220050176

Tian, G. (2015). K-stability and Kähler-Einstein metrics. *Comm. Pure Appl. Math.*, *68 (7)*, 1085–1156 pp.

Tian, G., & Yau, S.-T. (1983). On singularities on degenerate del Pezzo surfaces of degree 1, 2. *Proceedings of Symposia in Pure Mathematics*, *40 (2)*, 587-590 pp.

Tian, G., & Yau, S.-T. (1987). Kähler–Einstein metrics on complex surfaces with $c_1 > 0$. *Commun. Math. Phys.*, *112*, 175–203 pp.

Uhlenbeck, K., & Yau, S. T. (1986). On the existence of hermitian-yang-mills connections in stable vector bundles. *Communications on Pure and Applied Mathematics*, *39*, S257 – S293. doi: 10.1002/cpa.3160390714.

Wang, X. (2012). Height and GIT weight. *Math. Res. Lett.*, *19*(4), 909–926 pp.

Xu, C., & Zhuang, Z. (2020). On positivity of the cm line bundle on k-moduli spaces. *Annals of mathematics*, *192*(3), 1005–1068.

Yau, S.-T. (1977). Calabi's conjecture and some new results in algebraic geometry. *Proceedings of the National Academy of Sciences*, *74 (5)*, 1798-1799 pp. doi: 10.1073/pnas.74.5.1798

Yau, S.-T. (1978). On the Ricci curvature of a compact Kähler manifold and the complex monge-ampére equation, i. *Communications on Pure and Applied Mathematics*, *31 (3)*, 339–411 pp. doi: 10.1002/CPA.3160310304

Yau, S.-T. (1993). Open problems in geometry. *Differential Geometry: Partial Differential Equations on Manifolds (Los Angeles, CA, 1990). Proceedings of Symposia in Pure Mathematics*, *54*, 1–28 pp.

Zhuang, Z. (2021). Optimal destabilizing centers and equivariant K-stability. *Invent. Math.*, *226*, 195–223 pp.